Chapter 2 - Properties of Measures

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A. Restrictions of Measures

2.1 Definition. A pair (X, S) is called a measurable space if X is a set and S is a σ-ring of subsets of X. A triple (X, S, μ) is called a measure space if (X, S) is a measurable space and μ is a measure on S.

2.2 Definition. Let (X, S) be a measurable space. We say that $E \subseteq X$ is S-measurable (or just measurable) if $E \in S$. We say that $E \subseteq X$ is locally S-measurable if $E \cap F \in S$ for all $F \in S$. If μ is a non-negative measure on (X, S), then we say that $E \subseteq X$ is μ-measurable if $E \in S \oplus N(\mu)$ (which was defined in Proposition 1.41). We say that $E \subseteq X$ is locally μ-measurable if $E \cap F$ is μ-measurable for every μ-measurable set F.

We remark that if μ is σ-finite, then the μ-measurable sets are exactly the sets which are measurable with respect to the outer measure determined by μ, as is seen from Theorem 1.42.

It is easily seen that if (X, S) is a measurable space, then the family of all locally S-measurable sets is a σ-field, as is the family of all locally μ-measurable sets if μ is a non-negative measure on S.

If X is any set, P is a collection of subsets of X, and $E \subseteq X$, then by $P \cap E$ we will mean the collection $\{F \cap E : F \in P\}$. For example, if S is a σ-ring and E is a locally S-measurable set, then it is easily seen that $S \cap E$ will form a σ-ring which is contained in S. If μ is a measure on S, then we can obtain a measure on $S \cap E$ by restricting the domain of μ to $S \cap E$.

2.3 Definition. If μ is a measure on S and if E is a locally S-measurable set, then the measure obtained by restricting the domain of

μ to $S \cap E$ will be called <u>the restriction of μ to</u> E.

Given a measure μ' on $S \cap E$ we can enlarge its domain to obtain a measure μ on S by letting $\mu(F) = \mu'(F \cap E)$ for all $F \in S$. Note that if we start with μ on S, restrict μ to E and then enlarge back to a measure on S, we do not necessarily obtain μ back again. In particular, there will in general be many other ways of enlarging the domain of μ' to obtain a measure on S.

<u>2.4 Proposition</u>. If X is a set, P is a family of subsets of X and $E \subseteq X$, then $S(P \cap E) = S(P) \cap E$.

<u>Proof</u>. Since $S(P) \cap E$ is a σ-ring which contains $P \cap E$, it follows that $S(P \cap E) \subseteq S(P) \cap E$. We must show the reverse inclusion. Let T be the class of all sets of the form $A \oplus (B-E)$ where $A \in S(P \cap E)$ and $B \in S(P)$. Symbolically we may write $T = S(P \cap E) \oplus (S(P) \cap E')$. It is easy to verify that T is a σ-ring. If $F \in P$, then the relation $F = (F \cap E) \oplus (F-E)$ and the fact that $F \cap E \in P \cap E \subseteq S(P \cap E)$ show that $F \in T$, and therefore that $P \subseteq T$. It follows that $S(P) \subseteq T$, and therefore that $S(P) \cap E \subseteq T \cap E$. Since, however, it is clear that $T \cap E = S(P \cap E)$, it follows that $S(P) \cap E \subseteq S(P \cap E)$.//

For example, this proposition shows that the two natural ways of defining the Borel sets in the interval $[0, 1]$ coincide; we can either take the intersections with $[0, 1]$ of the Borel sets in $\mathbb{R}$, or we can apply directly to $[0, 1]$ the definition of the Borel sets of a topological space.

2.5 Corollary. If μ is a σ-finite premeasure on a semiring P, if $E \in P$ and if $\bar{\mu}$ is the extension of μ to $S(P)$ while $\hat{\mu}$ is the extension to $S(P \cap E)$ of μ restricted to $P \cap E$, then $\hat{\mu}$ is just the restriction of $\bar{\mu}$ to $S(P) \cap E$.

For example, this corollary shows that the two ways in which one might define Lebesgue measure on the interval $[0, 1)$ coincide.

B. The Total Variation of a Measure

In Chapter 1, most of the results which we proved involved non-negative measures. The purpose of this section is to define the total variation of an arbitrary measure. This gives us a way to obtain a non-negative measure which is closely related to a given arbitrary measure, and this will enable us to extend some of our earlier definitions and results about non-negative measures to arbitrary measures.

2.6 Definition. If μ is an arbitrary measure on a σ-ring S, then the total variation, $|\mu|$, of μ is defined by

$$|\mu|(E) = \sup\{ \sum_{i=1}^{n} \|\mu(E_i)\| : E = \bigoplus_{i=1}^{n} E_i, E_i \in S\}$$

for each $E \in S$. (Of course, if μ is extended real-valued, we let $\|\infty\| = \infty$.)

2.7 Theorem. The total variation, $|\mu|$, of a measure μ is a non-negative measure.

Proof. It is clear from its definition that $|\mu|$ is a non-negative extended real valued function, on S. We remark that it is also clear that $|\mu|$ is monotone, that is, that if $E \subseteq F$, where $E, F \in S$, then

$|\mu|(E) \leq |\mu|(F)$. To prove the theorem we need to show that $|\mu|$ is countably additive.

Suppose first that $E = F \oplus G$ with $F, G \in S$. If $F = \bigoplus_{i=1}^{m} F_i$ and $G = \bigoplus_{j=1}^{n} G_j$, then $E = \bigoplus_{i=1}^{m} F_i \oplus \bigoplus_{j=1}^{n} G_j$, and so

$$|\mu|(E) \geq \sum_{i=1}^{m} \|\mu(F_i)\| + \sum_{j=1}^{n} \|\mu(G_i)\|.$$

It follows that $|\mu|(E) \geq |\mu|(F) + |\mu|(G)$. By induction it follows that if $E = \bigoplus_{i=1}^{n} E_i$, then $|\mu|(E) \geq \sum_{i=1}^{n} |\mu|(E_i)$.

Suppose now that $E = \bigoplus_{i=1}^{\infty} E_i$. Then $E \supseteq \bigoplus_{i=1}^{n} E_i$ for all n, and so, since $|\mu|$ is monotone, $|\mu|(E) \geq |\mu|(\bigoplus_{i=1}^{n} E_i) = \sum_{i=1}^{n} |\mu|(E_i)$ for all n. Thus $|\mu|(E) \geq \sum_{i=1}^{\infty} |\mu|(E_i)$.

To prove the opposite inequality, suppose that $E = \bigoplus_{j=1}^{\infty} E_j$ and also that $E = \bigoplus_{i=1}^{n} F_i$. Then

$$\sum_{i=1}^{n} \|\mu(F_i)\| = \sum_{i=1}^{n} \|\mu(F_i \cap \bigoplus_{j=1}^{\infty} E_j)\| = \sum_{i=1}^{n} \|\mu(\bigoplus_{j=1}^{\infty} F_i \cap E_j)\|$$

$$= \sum_{i=1}^{n} \|\sum_{j=1}^{\infty} \mu(F_i \cap E_j)\| \leq \sum_{i=1}^{n} \sum_{j=1}^{\infty} \|\mu(F_i \cap E_j)\|$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{n} \|\mu(F_i \cap E_j)\| \leq \sum_{j=1}^{\infty} |\mu|(E_j),$$

since $E_j = \bigoplus_{i=1}^{n} (F_i \cap E_j)$ for each j. Thus $|\mu|(E) \leq \sum_{j=1}^{\infty} |\mu|(E_j)$, and so $|\mu|$ is countably additive.//

Note that $\|\mu(E)\| \leq |\mu|(E)$ for all $E \in S$, so that $|\mu|$ may be thought of as a non-negative measure which in a sense dominates μ. In exercise 1 at the end of this chapter you will be asked to show that $|\mu|$ is the smallest non-negative measure having this property.

As a result of Theorem 2.7 we can extend some of our earlier definitions which were originally made only for non-negative measures.

2.8 Definition. An arbitrary measure μ is said to be σ-finite (totally σ-finite) if $|\mu|$ is σ-finite (totally σ-finite).

2.9 Definition. If μ is an arbitrary measure on a σ-ring S, then a set $E \subseteq X$ is said to be (locally) μ-measurable if it is (locally) $|\mu|$-measurable (see Definition 2.2). A set $F \in H(S)$ is called a μ-null set (or just a null set) if $|\mu|^*(F) = 0$. We denote the family of μ-null sets by $N(\mu)$.

We remark again that Theorem 1.42 shows that if μ is σ-finite, then the μ-measurable sets are exactly the sets which are measurable with respect to the outer measure determined by $|\mu|$.

2.10 Definition. An arbitrary measure μ is said to be complete if every μ-null set is in the domain of μ.

The fact that we have used the term μ-measurable in Definition 2.9 suggests that we should be able to extend μ to a measure on the σ-ring of μ-measurable sets. The following theorem, which generalizes Proposition 1.41, shows that this is in fact the case.

2.11 Theorem. Let (X, S, μ) be an arbitrary measure space. Define $\hat{\mu}$ on the σ-ring of μ-measurable sets, $S \oplus N(\mu)$, by $\hat{\mu}(E \oplus F) = \mu(E)$ where $E \in S$ and $F \in N(\mu)$. Then $\hat{\mu}$ is a well defined complete measure which extends μ.

Proof. If $E \in S$ and $|\mu|(E) = 0$, then $\mu(E) = 0$. As a consequence, it is easily seen that the proof of Proposition 1.41 applies to this case also.//

C. Bounded Measures

2.12 Definition. A measure μ is said to be bounded if it does not take the value $+\infty$, that is, if all its values are in a Banach space. If μ is bounded and if in addition $X \in S$ (so that $\|\mu(X)\| < \infty$), then μ is said to be totally bounded.

We remark that a bounded measure need not be σ-finite. An example of such a measure will be given in exercise of Chapter 5.

The following proposition justifies the name "bounded".

2.13 Proposition. If μ is a bounded measure, then there exists a constant $K < \infty$ such that $\|\mu(E)\| < K$ for all $E \in$ domain μ.

Proof. Suppose not. Then for each n we can find a measurable set E_n such that $\|\mu(E_n)\| \geq n$. Let $E = \bigcup_{n=1}^{\infty} E_n$, so that $E \in S$. By construction, μ is unbounded on E, that is, for each $a \in \mathbb{R}$ there exists a measurable set $F \subseteq E$ such that $\|\mu(F)\| \geq a$. If the E_n were disjoint we would easily get a contradiction. We now construct three sequences of measurable sets, F_n, G_n and H_n, by induction, so that the G_n acts like the E_n but in addition are disjoint, and so that μ is unbounded on each H_n. Choose $F_1 \subseteq E$ such that $\|\mu(F_1)\| \geq \|\mu(E)\| + 1$. Then $\|\mu(F_1)\| \geq 1$ and $\|\mu(E-F_1)\| \geq 1$ (since $\|\mu(E-F_1)\| = \|\mu(E) - \mu(F_1)\| \geq |\|\mu(E)\| - \|\mu(F_1)\|| \geq 1$.) It is easily seen that μ must be unbounded on either F_1 or $E-F_1$ since otherwise it would not be unbounded on E. Let H_1 be one of F_1 or

$E-F_1$ in such a way that μ is unbounded on H_1, and let G_1 be the other of the two sets. We have thus defined F_1, G_1 and H_1. If F_{n-1}, G_{n-1} and H_{n-1} have been chosen, choose $F_n \subseteq H_{n-1}$ such that $\|\mu(F_n)\| \geq \|\mu(H_{n-1})\| + 1$. Then, as before, $\|\mu(F_n)\| \geq 1$, $\|\mu(H_{n-1} - F_n)\| \geq 1$, and μ is unbounded on either F_n or $H_{n-1} - F_n$. Let H_n be one of F_n or $H_{n-1} - F_n$ in such a way that μ is unbounded on H_n, and let G_n be the other of the two sets. The important thing to notice is that not only is $\|\mu(G_n)\| \geq 1$ for each n, but also the G_n are all disjoint. Let $G = \bigoplus_{n=1}^{\infty} G_n$. Then $\mu(G) = \sum_{n=1}^{\infty} \mu(G_n)$, and so $\sum_{n=1}^{\infty} \mu(G_n)$ is a series which converges to a finite (since μ is assumed bounded) value, but this is impossible since $\|\mu(G_n)\| \geq 1$ for each n.//

2.14 Definition. Let μ be a measure on a σ-ring S. A locally S-measurable set E is said to carry μ if $\mu(F) = \mu(F \cap E)$ for all $F \in S$, that is, if $F \cap E = \emptyset$ implies that $\mu(F) = 0$. We also sometimes say that μ lives on E.

2.15 Proposition. Let μ be a measure on a σ-ring S. If μ is a bounded measure, then there exists $E \in S$ on which μ lives.

Proof. By Proposition 2.13 we know that $\sup\{\|\mu(F)\| : F \in S\} < \infty$. We define a sequence, E_n, of elements of S by induction. Choose $E_1 \in S$ so that $\|\mu(E_1)\| \geq \frac{1}{2} \sup\{\|\mu(F)\| : F \in S\}$. If $E_1, \ldots, E_{n-1}$ have been chosen, choose E_n so that $E_n \cap \bigoplus_{i=1}^{n-1} E_i = \emptyset$ and $\|\mu(E_n)\| \geq \frac{1}{2} \sup\{\|\mu(F)\| : F \cap \bigoplus_{i=1}^{n-1} E_i = \emptyset,\ F \in S\}$. Let $E = \bigoplus_{n=1}^{\infty} E_n \in S$. Then $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n)$, so the series converge to a finite value, and so

$\|\mu(E_n)\|$ converges to 0 as n goes to ∞. We show that E carries μ. Suppose that $F \in S$ and $F \cap E = \emptyset$. Then $F \cap E_n = \emptyset$ for each n, and so, by the definition of the E_n, we have $\|\mu(E_n)\| \geq \frac{1}{2} \|\mu(F)\|$ for every n. Since the $\|\mu(E_n)\|$ converge to 0, it follows that $\mu(F) = 0$.//

In view of Proposition 2.15, it is natural to extend a bounded measure μ to a measure μ' on the σ-field of all locally measurable sets F by setting $\mu'(F) = \mu(F \cap E)$, where E carries μ. It is then easily seen that if μ is σ-finite so is μ'. Thus, in a sense, the only time we must work with σ-rings instead of σ-fields in order to preserve σ-finiteness is when we have an extended real valued measure.

If μ is a non-negative measure then it too can be extended to the σ-field of locally measurable sets F by letting $\mu'(F) = \sup\{\mu(E) : E \subseteq F, E \in \text{domain } \mu\}$. However, in general this μ' will not be σ-finite even if μ is σ-finite, and so it is usually not useful to make this extension.

D. Convergence Properties of Measures

The following two propositions will be very useful in later chapters.

2.16 Proposition. Let μ be a measure on a σ-ring S. If $\{E_n\}_{n=1}^{\infty}$ is a sequence of elements of S, and if $E_n \uparrow E$, that is, if $E_n \subseteq E_{n+1}$ for each n and $\bigcup_{n=1}^{\infty} E_n = E$, then $\mu(E_n)$ converges to $\mu(E)$ as n goes to ∞.

Proof. Clearly $E = E_1 \oplus \bigoplus_{i=1}^{\infty} (E_i - E_{i-1})$ and $E_n = E_1 \oplus \bigoplus_{i=1}^{n} (E_i - E_{i-1})$, and so

$$\mu(E) = \mu(E_1) \oplus \sum_{i=1}^{\infty} \mu(E_i - E_{i-1})$$
$$= \lim_n (\mu(E_1) + \sum_{i=1}^{n} \mu(E_i - E_{i-1})) = \lim_n \mu(E_n)$$

as desired.//

2.17 Proposition. Let μ be a measure on a σ-ring S. If $\{E_n\}_{n=1}^{\infty}$ is a sequence of elements of S such that $E_n \downarrow E$, that is, $E_n \supseteq E_{n+1}$ for each n and $\bigcap_{n=1}^{\infty} E_n = E$, and if $\|\mu(E_k)\| < \infty$ for some k, in case μ is an extended real-valued measure, then $\mu(E_n)$ converges to $\mu(E)$ as n goes to ∞.

Proof. We can assume that $\|\mu(E_1)\| < \infty$ since we can ignore a finite number of terms if we wish. But $(E_1 - E_n) \uparrow (E_1 - E)$, so by Proposition 2. $\mu(E_1 - E_n)$ converges to $\mu(E_1 - E)$ as n goes to ∞. Thus, since $\mu(E_1) - \mu(E_n) = \mu(E_1 - E_n)$ and $\mu(E_1) - \mu(E) = \mu(E_1 - E)$, we find that $\mu(E_1) - \mu(E_n)$ converges to $\mu(E_1) - \mu(E)$ as n goes to ∞. Since $\mu(E_1)$ is assumed finite, it follows that $\mu(E_n)$ converges to $\mu(E)$ as n goes to ∞.//

Note that Proposition 2.17 is not true without the hypothesis that $\|\mu(E_n)\| < \infty$ for some n. As an example, let μ be Lebesgue measure on $\mathbb{R}$ and let $E_n = [n, \infty)$.

Exercises

1. If μ is a vector valued measure, show that $|\mu|$ is the smallest extended real valued measure such that $\|\mu(E)\| \leq |\mu|(E)$ for all $E \in S$, that is, if ν is a non-negative measure on the domain of μ such that $\|\mu(E)\| \leq \nu(E)$ for all E in the domain μ, then $|\mu|(E) \leq \nu(E)$ for all E in the domain μ.

2. A measure μ is said to be of bounded variation if $|\mu|$ is a bounded measure. Show that any measure with values in a finite dimensional Banach space is of bounded variation. (You may assume that $B = \mathbb{R}^n$ with the usual Euclidean norm, since it can be shown that all norms on a finite dimensional vector space are equivalent, that is, any two norms, $\| \ \|$ and $\| \ \|_0$, satisfy $k\| \ \| \leq \| \ \|_0 \leq K\| \ \|$ for suitable constants k and K.)

3. Compute the total variations of those set functions in problem 1 of Chapter 1 which are measures. How does your result compare with problem 2 above?

4. Let B be a Banach space and let (X, S) be a measurable space. Then the collection of B-valued measures on S forms a vector space when $\mu+\nu$ is defined by $(\mu+\nu)(E) = \mu(E) + \nu(E)$ for all $E \in S$, and $\alpha\mu$ is defined by $(\alpha\mu)(E) = \alpha(\mu(E))$ for $E \in S$. (The field of scalars for the vector space of measures is taken to be the same as the field scalars for B. Note that we cannot form a vector space in this way if we admit measures taking the value $+\infty$.) Let M be the collection of all B-valued measures on S which are of bounded variation. It is easy to see that M is a subspace of the vector space of all B-valued measures on S. Furthermore, we can define a function $\| \ \|$ from M to $\mathbb{R}$ by

$$\|\mu\| = \sup\{|\mu|(E) : E \in S\}.$$

Show that $\|\ \|$ is a norm on M. (This is called the total variation norm.), and show that $(M, \|\ \|)$ is a Banach space. (Be sure to show that the set functions which you claim are measures really are.)

5. Two measures μ and ν on (X, S) are said to be <u>mutually singular</u> if there are disjoint locally measurable sets E, and F such that E carries μ and F carries ν.

   a) Show that if μ and ν are mutually singular then so are $|\mu|$ and $|\nu|$, and that if μ and ν have bounded variation, then $\|\mu+\nu\| = \|\mu\| + \|\nu\| = \|\mu-\nu\|$.

   b) Find Borel measures (i.e. measures on Borel sets) μ and ν on $[0, 1]$ which are mutually singular but are not carried on disjoint <u>closed</u> sets.