

Most of the following results are discussed in our recommended textbook by Davidson, but try obtaining them without looking at that book. You will find it more interesting and learn more that way. But if you get seriously stuck, you can look at that book.

Let $\theta \in \mathbb{R}$, and let α be the action of the group \mathbb{Z} on $C(\mathbb{T})$ by rotation by angle θ , and let $\mathcal{A}_\theta = C(\mathbb{T}) \times_\alpha \mathbb{Z}$, as discussed in class. When these “rotation algebras” first began to be studied, it was an open question, for the case in which θ is irrational, whether \mathcal{A}_θ contains any proper projections. After all, when $\theta = 0$ so that $\mathcal{A}_\theta \cong C(\mathbb{T}^2)$, it clearly does not contain any proper projections. But it turns out that in other cases \mathcal{A}_θ does contain proper projections. There is a simple but non-obvious construction for some of them:

1. Consider first the case in which $0 < \theta < 1/2$, and for each $n \in \mathbb{Z}$ let U_n be the corresponding unitary element in $\mathbb{Z} \subset C(\mathbb{T}) \times_\alpha \mathbb{Z} = \mathcal{A}_\theta$. Then consider elements of \mathcal{A}_θ of the form

$$P = fU_{-1} + gU_0 + hU_1,$$

where $f, g, h, \in C(\mathbb{T}) \subset C(\mathbb{T}) \times_\alpha \mathbb{Z}$. (View the functions in $C(\mathbb{T})$ as continuous functions on \mathbb{R} of period 1. You can assume f, g, h are real-valued if you wish.) Determine conditions on these functions such that P is a self-adjoint projection, i.e. $P^* = P$ and $P^2 = P$. Then exhibit specific relatively simple choices for these functions that satisfy the conditions, thus giving you specific proper projections in \mathcal{A}_θ .

2. When \mathcal{A}_θ is viewed as $C^*(\mathbb{Z}^2, c_\theta)$ one has the dual action of \mathbb{T}^2 on it. Determine the formula for this dual action when \mathcal{A}_θ is viewed as $C(\mathbb{T}) \times_\alpha \mathbb{Z}$. Then determine the formula for the unique tracial state that is invariant for this dual action, when \mathcal{A}_θ is viewed as $C(\mathbb{T}) \times_\alpha \mathbb{Z}$. Then calculate the value of this trace on the projection P which you constructed in part 1 above. Notice that $I - P$ is also a projection in \mathcal{A}_θ , and calculate its trace.

3. For any integer $k \geq 2$ we can restrict the action α to the subgroup $k\mathbb{Z}$ of \mathbb{Z} , and then construct the C*-algebra $C(\mathbb{T}) \times_\alpha k\mathbb{Z}$, and then use the methods of part 1 to construct a projection, and then calculate its trace as in part 2. (Notice that this in effect uses $k\theta$ but that it is only the values modulo 1 that are involved, so that one still uses “angles” in the interval $(0, 1)$). But $C(\mathbb{T}) \times_\alpha k\mathbb{Z}$ is clearly a subalgebra of $C(\mathbb{T}) \times_\alpha \mathbb{Z}$, and so in this way we obtain many more projections in \mathcal{A}_θ . What are the various values of the trace on projections that we obtain in this way?

4. Also, for any integer $k \geq 2$ we can consider the algebra $C(\mathbb{T}/k)$ of continuous functions on \mathbb{R} of period $1/k$, which is a subalgebra of $C(\mathbb{T})$. It is

carried into itself by α , and so we can construct the algebra $C(\mathbb{T}/k) \times_{\alpha} \mathbb{Z}$, which is clearly a subalgebra of $C(\mathbb{T}) \times_{\alpha} \mathbb{Z}$. So again we can use the methods of part 1 to construct projections in this algebra and calculate their traces. So in this way we obtain many more projections in \mathcal{A}_{θ} . What are the various values of the trace on projections that we obtain in this way? Do you see a pattern? (But proving that all projections fit the pattern is hard.)

5. Two projections p and q in an algebra \mathcal{A} are said to be (*von Neumann*) *equivalent* if there are elements u and v in \mathcal{A} such that $p = uv$ and $q = vu$. Notice that this is an equivalence relation on projections, and that any trace will have the same value on two equivalent projections. It is an important fact that if p and q are two (self-adjoint) projections in a C^* -algebra \mathcal{A} such that $\|p - q\| < 1$ then they are equivalent (and that one can find u such that $v = u^*$). I am not asking you to prove this, though a proof can be obtained by carefully examining the answer to problem 4 of problem set 5 for this course, though there are simpler proofs. But use this fact to show that if \mathcal{A} is a C^* -algebra that is separable, then for any given trace on \mathcal{A} the collection of all its values on all projections in \mathcal{A} is a countable set of positive real numbers. Conclude from this that among all of the \mathcal{A}_{θ} 's for different θ 's there is it an uncountable number of distinct isomorphism classes of algebras. (But one needs some more information if one wants to exhibit two specific θ 's for which the algebras are not isomorphic).

[A function defined on an open interval of \mathbb{R} is said to be *analytic* if it has a power-series expansion of > 0 radius of convergence about each point in its domain. Then by “analytic continuation” such a function is determined by its values on any small interval in its domain. Notice that your construction in part 1 will not work if the functions f, g, h are required to be analytic. A consequence of this is that in the purely algebraic algebra $C_c(\mathbb{Z}^2, c_{\theta})$ one will not find versions of the projections of part 1. In fact, I believe this algebra contains no proper projections, though at the moment I do not remember a reference for that. It is only by at least partially completing, by using $C(\mathbb{T})$, that the projections appear. Smooth functions, i.e. infinitely differentiable functions will work.]