M. Rieffel Math 208, Problem Set 6 due Friday April 10, 2026

1. In preparation for problem 2 below, it is useful to note that for actions of groups on topological spaces, not only closed orbits are favorable, but also orbits that are open in their closure are favorable. As background we mention that it can be shown that a subset A of a locally compact Hausdorff space X will be locally compact for the relative topology if and only if it is open in its closure in X. (You can have fun trying to prove that if you wish.)

Consider the following example. Let $X = \mathbb{R} \cup \{+\infty\}$ with its usual topology, so that $C_\infty(X)$ consists of the continuous functions on $\mathbb{R}$ that vanish at $-\infty$ and have a limit at $+\infty$. Let α be the action of $G = \mathbb{R}$ on X that acts on the subset $\mathbb{R}$ by translation and leaves the point $+\infty$ fixed. Note that this action is jointly continuous. Let $\mathcal{A} = C_\infty(X) \times_\alpha G$, the corresponding covariance (or "crossed-product")algebra. Determine the irreducible representations of $\mathcal{A}$, and the topology on the corresponding primitive ideal space. Is this algebra CCR or GCR?

2. For any $n \times n$ real matrix T define an action α of $\mathbb{R}$ on the group $\mathbb{R}^n$ by $\alpha_t = \exp(tT)$ acting in the evident way. Let $G = \mathbb{R}^n \times_\alpha \mathbb{R}$, a semi-direct product. (Then G is a solvable Lie group.) For the case of $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ determine the equivalence classes of irreducible unitary representations of G, i.e. the irreducible representations of $C^*(G)$. Determine the topology on the primitive ideal space $\mathrm{Prim}(C^*(G))$. Draw a picture. To describe the topology it is useful to chose a "fundamental domain", that is, a choice of a subset consisting of one point from each orbit, chosen in a particularly convenient way, and then describing the topology on that set. Discuss whether $C^*(G)$ is CCR or GCR, and why. (Hint: As suggested briefly in class, you will find the Fourier transform on $\mathbb{R}^2$ useful.)

(As a "warm-up exercise", you might try first the simpler case for which $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.)

These examples can give you a hint of the complexities of the representation theory of Lie groups, and of some of the techniques used to deal with them.