M. Rieffel   Math 208, Problem Set 5   due Monday March 30, 2026

In my Friday lecture about defining C*-algebras by generators and relations, I did not get to say that the corresponding C*-algebra is often called the "universal" C*-algebra for the given generators and relations. And the relations are often stated by saying what properties the generators should have for representations of the algebra on Hilbert spaces.

Also, for a dynamical system $(\mathcal{A}, G, \alpha)$, the corresponding covariance algebra $C^*(\mathcal{A}, G, \alpha)$ is often denoted by $A \times_\alpha G$.

1. Show that the following C*-algebras are isomorphic:

a) The universal unital C*-algebra generated by two (self-adjoint) projections.

b) The universal C*-algebra generated by two self-adjoint unitary elements.

c) The group algebra $C^*(G)$ for $G = \mathbb{Z}_2 * \mathbb{Z}_2$, the free product of two copies of the 2-element group.

d) The crossed-product algebra $A \times_\alpha G$ where $A = C(T)$ for T the unit circle in the complex plane, $G = \mathbb{Z}_2$, and α is the action on T of taking complex conjugation. (So T/α exhibits the unit interval as an "orbifold", i.e. the orbit-space for the action of a finite group on a manifold, and $A \times_\alpha G$ remembers where the orbifold comes from.) Hint: In $\mathbb{Z}_2 * \mathbb{Z}_2$ find a copy of $\mathbb{Z}$.

2. Use the center of $A \times_\alpha G$ to express $A \times_\alpha G$ as a continuous field of C*-algebras.

3. Determine the primitive ideal space of $A \times_\alpha G$, with its hull-kernel topology.

4. Identify the two elements of $A \times_\alpha G$ that correspond to the two projections of part a).