Fields of C^*-algebras.   Anytime the center of a C^*-algebra (i.e. the set of its elements which commute with all the elements of the algebra) is more than one-dimensional and acts non-degenerately on the algebra, the C^*-algebra can be decomposed as a field of C^*-algebras over the maximal ideal space of the center (or of any non-degenerate C^*-subalgebra of the center). This can give important insight into the structure of a C*-algebra, as we will see on future problem sets.

For simplicity we deal here with unital algebras (for which the center always acts non-degenerately), but all of this works without difficulty in general. So let A be a C^*-algebra with 1, and let C be a C^*-subalgebra of the center of A with $1 \in C$. Let $C \cong C(X)$, and for $x \in X$ let J_x be the ideal of functions vanishing at x. Let $I_x = AJ_x$ (closure of linear span), a two-sided ideal in A. Let $A_x = A/I_x$ ("localization"), so that $\{A_x\}_{x \in X}$ is a "field" of C^*-algebras over X. For $a \in A$ let a_x be its image in A_x.

1) Prove that for any $a \in A$ the function $x \mapsto \|a_x\|_{A_x}$ is upper-semi-continuous. (So $\{A_x\}$ is said to be an upper-semi-continuous field.)

2) If $x \mapsto \|a_x\|_{A_x}$ is continuous for all $a \in A$, then the field is said to be continuous. For this part assume that A is commutative. Note that then one gets a continuous surjection from $\hat{A}$ onto $\hat{C}$. Find examples of A's and C's for which $x \mapsto \|a_x\|$ is not continuous. In fact, find an attractive characterization of exactly when the field is continuous, in terms of the surjection from $\hat{A}$ onto $\hat{C}$ and concepts that you have probably met in the past. (It can be shown that an analogous characterization works in the non-commutative case, using the primitive ideal space, see part 4 below, of A.) Hint: Try various examples involving compact subsets of the plane and their projections to the x-axis.

3) Consider the C*-algebras

$$A_1 \quad = \{f : [0, 1] \to M_2 \text{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\}$$

$$A_2 \quad = \{f : [0, 1] \to M_2 \text{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\}$$

$$A_3 \quad = \{f : [0, 1] \to M_3 \text{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & \delta & \epsilon \end{pmatrix}\},$$

for some $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}$ depending on f. Let $C = \text{center}(A)$ in each case. Are the corresponding fields continuous? Are all the fiber algebras A_x isomorphic? Show that A_1 and A_2 are not isomorphic. The above A_1, A_2 and A_3 are very simple prototypes of behavior that occurs often "in nature", but with higher-dimensional algebras, and more complicated boundary behavior, as we will see later in the course.

4) By definition a "primitive ideal" in a C*-algebra is the kernel of an irreducible $*$-representation of the algebra. The set of primitive ideals of a

C*-algebra A is called its "primitive ideal space", and is often denoted by $\hat{A}$. Determine the primitive ideal space of each of the three algebras given in part 3). Hint: So you will need to determine their irreducible representations. For this remember Schur's Lemma, and consider its implications for the center of the algebra.

5) For any C*-algebra A there is a natural topology, the "hull-kernel topology", on $\hat{A}$. Given a subset $S \subseteq \hat{A}$, its kernel (a quite different meaning for this word), $k(S)$, is by definition the ideal that is the intersection of all the elements of S. (It may be $\{0\}$.) Given any (2-sided closed) ideal J in A, its hull, $h(J)$, is by definition the set of all elements of $\hat{A}$ that contain J. The closure of a subset $S \subseteq \hat{A}$ is by definition $h(k(S))$. This determines the closed sets for a topology. (You are not asked to prove this, though it is not very difficult, and is closely related to a corresponding topology on the space of prime ideals in a unital ring, because every primative ideal is prime. That topology on the prime ideals is called the Jacobson topology, and if the ring is commutative it is called the Zariski topology.) Determine the hull-kernel topology on $\hat{A}$ for the three algebras of part 3.