M. Rieffel Math 208 Problem Set 3
due Friday February 27, 2026

Compactifications, and the non-commutative Stone-Cech compactification. The results of these problems are quite important for dealing with non-unital C*-algebras, viewed as "non-commutative (algebras of functions on) locally compact (Hausdorff) spaces". These problems are also a good exercise in working with normed algebras and C*-algebras. At a few points later in the course we will need to use results from some of these problem. So remember the main results. Even for ordinary locally compact spaces the results can be useful, because it is common to need a compactification, often specified by requiring certain "behavior at infinity", and this can be conveniently made precise by the methods given below.

Motivation: If the locally compact (Hausdorff) space X is an open subset of the compact space Y, then $C_\infty(X)$ "is" an ideal in the C*-algebra $C_b(Y)$ of bounded continuous functions on Y, by extending functions on X to have value 0 off of X. Then X is dense in Y exactly if $C_\infty(X)$ is an essential ideal in $C_b(Y)$, where by definition, a (two-sided) ideal I in an algebra B is **essential** if there is no non-zero ideal J in B such that $IJ = 0$ or $JI = 0$. If X is dense in Y, then Y is said to be a "compactification" of X. The Stone-Cech compactification of a locally compact space is the largest compactification in a specific sense (suggested by problem 5 below). Thus the Stone-Cech compactification of an algebra A without 1 should be a "maximal" algebra with 1 in which A sits as an essential two-sided ideal. If B is any algebra in which some (probably non-unital) algebra A sits as a two-sided ideal, then each $b \in B$ defines a pair (L_b, R_b) of operators on A defined by $L_b a = ba$, $R_b a = ab$. These operators satisfy, for $a, c \in A$, $L_b(ac) = (L_b(a))c$, $R_b(ca) = c(R_b a)$, and $a(L_b c) = (R_b a)c$.

Definition: By a **double centralizer** (or **multiplier**) on an algebra A (no norm assumed) we mean a pair (S, T) of operators on A satisfying the above three conditions. Let $M(A)$ denote the set of double centralizers of A.

1. Using the example of A as an ideal in B as motivation, define operations on $M(A)$ making it into an algebra, with a homomorphism of A onto an ideal of $M(A)$. (To hand in, just write the formulas for the addition and multiplication on $M(A)$, and for the homomorphism of A into $M(A)$. You don't need to write out the verification that they work, but you should have checked that carefully.)

2. Show that if A is a Banach algebra with two-sided approximate identity of norm one, and if we require S and T to be continuous (which actually is automatic), then $M(A)$ can be made into a Banach algebra in which A sits isometrically as an essential ideal. (This is quite useful for various Banach algebras which are not C^*-algebras. For example, once we have defined the convolution algebra $A = L^1(G)$ for a locally compact group G, it can be shown that $M(A)$ is the convolution measure algebra $M(G)$ of G.) Show that if A is a Banach $*$-algebra, then its involution extends uniquely to make $M(A)$ a Banach $*$-algebra. (For the involution just write the formula, but be sure to have checked that your formula really works. It is a bit tricky.)

3. Prove that if A is a C^*-algebra, then so is $M(A)$.

4. Let A be a C^*-algebra, and let $V = A$ viewed as a right A-module, with A-valued inner product as defined in class. Let $B_A(V)$ be the algebra of all continuous (which actually is automatic) A-module endomorphisms of V that have a continuous endomorphism as adjoint for the A-valued inner product (which is not automatic, in contrast to ordinary Hilbert spaces). Show that in a very natural way $M(A) = B_A(V)$.

5. For A a C^*-algebra, show that if B is any unital C^*-algebra in which A sits as an essential ideal, then B can be identified as a unital subalgebra of $M(A)$. Thus $M(A)$ is maximal in this sense, and thus can be considered to be the Stone-Cech compactification of A. (Such other unital C^*-algebras B are viewed as smaller compactifications of A. For various specific examples

they can be important, just as in the commutative case, i.e. for locally compact spaces. The smallest compactification is obtained just by adjoining an identity element. This corresponds to the "one-point compactification" from general topology.)

6. Determine (i.e. find a concrete realization of) $M(A)$ when:

a) $A = C_\infty(X)$ for a locally compact space X.

b) $A = B_o(\mathcal{H})$, the algebra of compact operators on a Hilbert space $\mathcal{H}$. (So $B_o(\mathcal{H})$ is the norm-closure of the ideal of finite-rank operators on H.)

Give careful proofs that your answers are correct.

$*$) If you would like a good example to chew on for ordinary spaces, here is one, but I am not asking you to do it.

By now you have all the tools needed to do this problem. But you should not be surprised if it takes you a considerable amount of methodical thought to see what is going on. But there is a nice answer.

Let $\mathcal{C}_b(\mathbb{R})$ be the C*-algebra of bounded continuous $\mathbb{C}$-valued functions on $\mathbb{R}$. Let $\mathcal{A}$ be the C*-subalgebra of $\mathcal{C}_b(\mathbb{R})$ generated by $\mathcal{C}_\infty(\mathbb{R})$ (the subalgebra of functions that "vanish at infinity") together with the function $f(t) = e^{it}$. Determine the "maximal ideal space", $\hat{\mathcal{A}}$, of $\mathcal{A}$, e.g. describe a subset of $\mathbb{R}^n$ for some n that, in a relatively simple way you can explain, is homeomorphic to $\hat{\mathcal{A}}$. A somewhat carefully labeled and explained drawing may be a good way to present your answer (and it may remind you of a somewhat well-known toy, pictured on Wikipedia). You will see, as expected, that $\hat{\mathcal{A}}$ is a compactification of $\mathbb{R}$, reflecting the fact that $\mathcal{C}_\infty(\mathbb{R})$ is an essential ideal in $\mathcal{A}$.