

Let  $\mathcal{A} = \mathcal{B}(\mathcal{H})$  for a Hilbert space  $\mathcal{H}$  over  $\mathbb{C}$ , and let  $\mathcal{A}^{sa}$  be its subset of self-adjoint elements, which is a closed  $\mathbb{R}$ -subspace of  $\mathcal{A}$ . There is a very important partial order on  $\mathcal{A}^{sa}$ , determined by saying, for any  $T \in \mathcal{A}^{sa}$ , that  $T \geq 0$  if  $\langle T\xi, \xi \rangle \geq 0$  for all  $\xi \in \mathcal{H}$ . Then for  $S, T \in \mathcal{A}^{sa}$  we say that  $S \geq T$  if  $S - T \geq 0$ . (This does not work if  $\mathcal{H}$  is over  $\mathbb{R}$ . For  $\mathcal{H} = \mathbb{R}^2$  with its usual inner product, consider the operator of rotation by 90 degrees.) This gives a very important order on any  $C^*$ -subalgebra of  $\mathcal{A}$ .

Some of the properties of this order on  $C^*$ -algebras are anti-intuitive. This is probably related to the anti-intuitive phenomena found in quantum mechanics. These anti-intuitive properties already show up in the  $C^*$ -algebra  $M_2(\mathbb{C})$ . It is easy to make errors when using the order properties. Thus:

Preventive Medicine:

Note that a self-adjoint matrix  $A$ , it is positive if all its eigenvalues are non-negative. For  $A \in M_2(\mathbb{C})$  this is the same as saying the diagonal entries are non-negative and the determinant is non-negative.

Let  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$  for some  $s, t \in \mathbb{R}$ , viewed as elements of  $M_2(\mathbb{C})$ .

- a) Determine for which  $s, t$  we have  $B \geq A$ .
- b) Determine for which  $s, t$  we have  $B \geq A^+$  (the positive part of  $A$ , which can be defined in general by considering the commutative  $C^*$ -subalgebra generated by  $A$ ).
- c) Find values of  $s, t$  such that  $B \geq A$  and  $B \geq 0$ , and yet it is false that  $B \geq A^+$ . (So be careful about making false proofs, and see part i) below.)
- d) Find values of  $s, t$  such that  $B \geq A^+$  and yet it is false that  $B^2 \geq (A^+)^2$ . (So again be careful about making false proofs.)
- e) Can you find values of  $s, t$  such that  $B \geq A^+$  and yet it is false that  $B^{1/2} \geq (A^+)^{1/2}$ ?
- f) For  $2 \times 2$  matrices  $T$  and  $P$  such that  $T \geq 0$  and  $P$  is an orthogonal projection (i.e.  $P^2 = P = P^*$ ), is it always true that  $PTP \leq T$ ?

In the literature there are many important unobvious order inequalities involving operators on Hilbert spaces. For example, there is much interesting information about which continuous functions  $f$  from  $\mathbb{R}^+$  to itself have the property that whenever  $A$  and  $B$  are operators such that  $A \geq B \geq 0$  one then has  $f(A) \geq f(B) \geq 0$ .

g) For each  $r \in \mathbb{R}$  with  $r > 0$  set  $S_r = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ . Show that for each  $r$  we have  $S_r \geq A$ . Thus  $S_r$  is an upper bound for  $A$  and  $0$  for each  $r$ .

h) Determine for which values of  $r$  we have  $S_r \geq A^+$ . Thus notice that although  $A^+$  is the least upper bound for  $A$  and  $0$  in the commutative subalgebra generated by  $A$ , it is not a least upper bound for them within  $\mathcal{A}^{sa}$ .

i) Suppose that  $D$  is a least upper bound for  $A$  and  $0$  within  $\mathcal{A}^{sa}$ . Then it must satisfy  $D \geq A$  and  $D \leq S_r$  for all  $r$ . Prove that no such  $D$  exists. Thus  $\mathcal{A}^{sa}$  is not lattice ordered. (This result is easily extended to  $\mathcal{A} = M_n(\mathbb{C})$  for  $n > 2$ .) But it is important to note that in any C\*-algebra  $\mathcal{A}$ , any  $A \in \mathcal{A}^{sa}$  has a “Jordan decomposition”  $A = A^+ - A^-$  with  $A^+ \geq 0$ ,  $A^- \geq 0$  and  $A^+A^- = 0$ , with both  $A^+$  and  $A^-$  commuting with  $A$ , as seen by considering the commutative C\*-subalgebra generated by  $A$ .

[It is a theorem that for any C\*-algebra  $\mathcal{A}$  we have  $\mathcal{A}^{sa}$  is lattice ordered if and only if  $\mathcal{A}$  is commutative.]