1. Let $G = \mathbb{Z}_3$, the finite cyclic group of order 3.

a) Let $\mathcal{A} = \ell^1(G)$ for $\mathbb{C}$-valued functions, with convolution as product. Determine the maximal ideals of $\mathcal{A}$. How many of them are there? For each of them identify the quotient field A/I as a familiar field.

b) Instead, let $\mathcal{A} = \ell^1_{\mathbb{R}}(G)$ for *real*-valued functions, with convolution as product. Determine the maximal ideals of $\mathcal{A}$. How many of them are there? (Be careful. Prove that the ideals you obtain are distinct.) What are their dimensions? Identify the corresponding quotient fields. Can several different homomorphisms into a field have the same kernel? This illustrates that bookkeeping for commutative Banach algebras can be more complicated when working over $\mathbb{R}$. How are this part and part a) related to representations of G, i.e. homomorphisms of G into the group of invertible operators on a vector space?

c) For $\mathcal{A} = \ell^1(G)$ for $\mathbb{C}$-valued functions, and equipped with its ℓ^1 norm, show that its Gelfand transform is not isometric. This shows, in particular that $\mathcal{A}$ with its ℓ^1 norm is not a C*-algebra.

Contemplate what happens for the above questions for $G = \mathbb{Z}_n$ when $n > 3$. (The “filters” in your digital cell-phones, MP3-players, etc, are often convolutions by functions in $\ell^1_{\mathbb{R}}(G)$ for $G = \mathbb{Z}_n$ with $n > 1000$.)