Fields of $C^*$-algebras. Anytime the center of a $C^*$-algebra (i.e. the set of its elements which commute with all the elements of the algebra) is more than one-dimensional and acts non-degenerately on the algebra, the $C^*$-algebra can be decomposed as a field of $C^*$-algebras over the maximal ideal space of the center (or of any non-degenerate $C^*$-subalgebra of the center). For simplicity we deal here with unital algebras, but all of this works without difficulty in general. So let $A$ be a $C^*$-algebra with 1, and let $C$ be a $C^*$-subalgebra of the center of $A$ with $1 \in C$. Let $C = C(X)$, and for $x \in X$ let $J_x$ be the ideal of functions vanishing at $x$. Let $I_x = AJ_x$ (closure of linear span), an ideal in $A$. Let $A_x = A/I_x$ (“localization”), so that $\{A_x\}_{x \in X}$ is a “field” of $C^*$-algebras over $X$. For $a \in A$ let $a_x$ be its image in $A_x$.

1) Prove that for any $a \in A$ the function $x \mapsto \|a_x\|_{A_x}$ is upper-semi-continuous. (So $\{A_x\}$ is said to be an upper-semi-continuous field.)

2) If $x \mapsto \|a_x\|_{A_x}$ is continuous for all $a \in A$, then the field is said to be continuous. For this part assume that $A$ is commutative. Note that then one gets a continuous surjection from $\hat{A}$ onto $\hat{C}$. Find examples of $A$’s and $C$’s for which $x \mapsto \|a_x\|$ is not continuous. In fact, find an attractive characterization of exactly when the field is continuous, in terms of the surjection from $\hat{A}$ onto $\hat{C}$ and concepts that you have probably met in the past. (It can be shown that an analogous characterization works in the non-commutative case, using the primitive ideal space, see part 4 below, of $A$.) Hint: Try various examples involving compact subsets of the plane and their projections to the x-axis.

3) Consider the $C^*$-algebras

$$A_1 = \{f : [0, 1] \to M_2 \text{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \}$$

$$A_2 = \{f : [0, 1] \to M_2 \text{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \},$$

and let $C = \text{center}(A)$ in each cases. Are the corresponding fields continuous? Are all the fiber algebras $A_x$ isomorphic? Show that $A_1$ and $A_2$ are not isomorphic. The above $A_1$ and $A_2$ are very simple prototypes of behavior that occurs often “in nature”, but with higher-dimensional algebras, and more complicated boundary behavior.

4. Show that the following $C^*$-algebras are isomorphic:

1. The universal unital $C^*$-algebra generated by two (self-adjoint) projections.

2. The universal $C^*$-algebra generated by two self-adjoint unitary elements.

3. The group algebra $C^*(G)$ for $G = \mathbb{Z}_2 * \mathbb{Z}_2$, the free product of two copies of the 2-element group.
4. The crossed-product algebra $A \times_\alpha G$ where $A = C(T)$ for $T$ the unit circle in the complex plane, $G = \mathbb{Z}_2$, and $\alpha$ is the action of taking complex conjugation. (So $T/\alpha$ exhibits the unit interval as an “orbifold”, i.e. the orbit-space for the action of a finite group on a manifold, and $A \times_\alpha G$ remembers where the orbifold comes from.) Hint: In $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$ find a copy of $\mathbb{Z}$.

5. Determine the primitive ideal space of the above C*-algebra, with its topology.

6. Use the center of the algebra above to express the algebra as a continuous field of C*-algebras.

7. Use problem 6 to prove that if $p$ and $q$ are two projections in a unital C*-algebra such that $\|p - q\| < 1$, then they are unitarily equivalent, that is, there is a unitary element $u$ in the algebra (in fact in the unital subalgebra generated by $p$ and $q$) such that $upu^* = q$.

8. Use problem 7 to show that in a unital separable C*-algebra the set of unitary equivalence classes of projections is countable.