1. Let $X$ and $Y$ be compact Hausdorff spaces, and let $F$ be a continuous function from $X$ to $Y$.

a) Define a function $\Phi_F$ from $C(Y)$ to $C(X)$ by $\Phi_F(f) = f \circ F$. Show that $\Phi_F$ is a unital $*$-homomorphism from $C(Y)$ to $C(X)$ (where this means that $\Phi_F$ is an algebra homomorphism that carries identity element to identity element, and respects complex conjugation).

b) Show that the mapping $F \mapsto \Phi_F$ is a bijection between the set of continuous functions from $X$ to $Y$ and the set of unital $*$-homomorphisms from $C(Y)$ to $C(X)$.

c) Show that if also $G$ is a continuous function from $Y$ to a compact Hausdorff space $Z$, then $\Phi_{G \circ F} = \Phi_F \circ \Phi_G$. (Of course the usual associative law for compositions then holds here. One then says, by definition, that the process that for each $X$ sends it to $C(X)$ and that for each $F$ sends it to $\Phi_F$ is a “functor” from the category of compact Hausdorff spaces and continuous functions between them, to the category of unital commutative $C^*$-algebras with continuous unital homomorphisms between them. This functor is “contravariant” because it reverses the order of $X$ and $Y$. For any category, its abstract dual is the category with the same objects, but with the “arrows”, i.e. “morphisms” turned around, so that the evident “identity functor” from the category to its dual is contravariant. Because we now know that every unital commutative $C^*$-algebra is naturally isomorphic to a $C(X)$ for some compact Hausdorff space, this part c) together with part b) above completes the proof that the category of unital commutative $C^*$-algebras with unital $*$-homomorphisms between them is a “concrete realization” of the dual of the category of compact Hausdorff spaces with continuous functions between them. Thus in principle anything that can be done for compact Hausdorff spaces has a version for unital commutative $C^*$-algebras and conversely.)

2. Let $C_b(\mathbb{R})$ be the $C^*$-algebra of bounded continuous $\mathbb{C}$-valued functions on $\mathbb{R}$. Let $\mathcal{A}$ be the $C^*$-subalgebra of $C_b(\mathbb{R})$ generated by $C_\infty(\mathbb{R})$ (the subalgebra of functions that “vanish at infinity”) together with the function $f(t) = e^{it}$. Determine $\hat{\mathcal{A}}$, e.g. describe a subset of $\mathbb{R}^n$ for some $n$ that in a relatively simple way you can explain is homeomorphic to $\hat{\mathcal{A}}$. A somewhat carefully labeled and explained drawing may be a good way to present your answer (and it may remind you of a somewhat well-known toy, pictured on Wikipedia). You will find that $\hat{\mathcal{A}}$ is a compactification of $\mathbb{R}$, with $\mathbb{R}$ being an open and dense subset. This reflects the fact that $C_\infty(\mathbb{R})$ is an ideal in $\mathcal{A}$. Hint: Exploit this ideal, and think quite carefully about how everything is related.