1. Let $T = \{ z \in \mathbb{C} : |z| = 1 \}$, the unit circle. It is a topological group under multiplication. Define the group homomorphism $e : \mathbb{R} \to T$ by $e(r) = \exp(2\pi ir)$. Let $\mu$ be Lebesgue measure on $\mathbb{R}$ with $M$ the collection of Lebesgue measurable sets. Set $M(T) = \{ E \subseteq T : e^{-1}(E) \in M \}$. Show that $M(T)$ is a $\sigma$-ring. Define $\nu$ on $M(T)$ by $\nu(E) = \mu(e^{-1}(E) \cap [0,1))$. Show that $\nu$ is a measure, and that $\nu$ is “translation-invariant”, i.e. “rotation-invariant”, in the evident sense.

2. Let $\alpha$ be the non-decreasing function on $\mathbb{R}$ defined by $\alpha(t) = 0$ if $t \leq 0$ and $\alpha(t) = 1$ if $t > 0$. Let $\mu_\alpha$ be the corresponding premeasure as discussed in lecture, with $\mu^*_\alpha$ the corresponding outer measure. Determine which sets are in $M(\mu^*_\alpha)$, that is, measurable for this outer measure.

3. Let $\alpha$ be the function on $\mathbb{R}$ defined by $\alpha(r) = r$, which is the function that leads to Lebesgue measure on $\mathbb{R}$. Let $\mu_\alpha$ be the corresponding premeasure, and let $\mu^*_\alpha$ be the corresponding outer measure on $\mathbb{R}$. Let $Q$ be the set of rational numbers, viewed as a subset of $\mathbb{R}$. Calculate $\mu^*_\alpha(Q)$. 