1. Given two sets, $X$ and $Y$, and a function, $f$, from $X$ to $Y$, the graph of $f$, denoted $\text{Gr}(f)$, is the subset $\{(x, f(x)) : x \in X\}$ of $X \times Y$. Suppose now that $X$ and $Y$ are topological spaces, and that $Y$ is Hausdorff. Prove that if $f$ is continuous then its graph is closed for the product topology on $X \times Y$. Give an example of an $\mathbb{R}$-valued function on $\mathbb{R}$ which is not continuous and yet whose graph is closed.

2. For any given positive integer $p$ let $S_p = \{1, 2, \cdots, p\}$, the “set of symbols”. For each $k \in \mathbb{Z}$ let $X_k = S_p$, and form the product space $Y_p = \prod_{k=-\infty}^{\infty} X_k$. It thus consists of all functions from $\mathbb{Z}$ to $S_p$. Equip $S_p$, and so each $X_k$ with the discrete topology, and then equip $Y_p$ with the corresponding product topology.

   a) Let $B$ denote the usual base for the product topology, for the case of the above $Y_p$. Show that each element of $B$ is actually closed. (In this context the elements of the usual base are often called “cylinder sets”. A subset of a topological space is said to be “clopen” if it is both open and closed. A topological space whose topology has a base consisting entirely of clopen sets is said to be “0-dimensional”. Thus what you are being asked to show is that the above space $Y_p$ is 0-dimensional.)

   b) Define a function, $T$, from $Y_p$ to itself by $(Tx)_k = x_{k-1}$ for each $x \in Y_p$. Show that $T$ is a homeomorphism of $Y_p$ with itself. (Usually $T$ is called the “right shift” on $Y_p$. In some contexts $T$ is called the “full Bernoulli shift”. A movie recorded on a DVD disk consists of a very long string of 0’s and 1’s, corresponding to pits burned in the DVD disk. It is useful to consider these strings to be infinitely long, and so to be elements of $X_2$ (using 1’s and 2’s instead of 0’s and 1’s). This view is useful in considering different schemes for compression of data, etc. In probability theory $Y_2$ is useful in considering “coin-tossing” questions, often with a probability “measure” on $S_p$ and a corresponding measure on $Y_2$.)

   c) For each positive integer $n$ we let $T^n$ be the composition of $T$ with itself $n$ times. In the same way we define $T^n$ for negative $n$ by using $T^{-1}$. Show that the function $n \mapsto T^n$ is a group homomorphism of $\mathbb{Z}$ into the group of homeomorphisms from $Y_p$ to itself. In this way we define an “action” of $\mathbb{Z}$ on $Y_p$. (Very often the integers are viewed as discrete time, the “ticks of the clock”, and the system consisting of $Y_p$ and the action of $\mathbb{Z}$ given by $T$ is considered to be a “discrete” dynamical system.)
d) Given the above action of \( \mathbb{Z} \) on \( Y_p \), we can consider the orbits of points in \( Y_p \). For any \( x \in Y_p \) its orbit is \( \{ T^n(x) : n \in \mathbb{Z} \} \). We will soon see in class that \( Y_p \) is compact, and so orbits are likely to have limit points. Let \( x \) be the specific element of \( Y_2 \) defined by \( x_1 = 2, x_3 = 2, \) and \( x_k = 1 \) if \( k \neq 1 \) and \( k \neq 3 \). Determine the closure of the orbit of this \( x \).

e) Find a nice characterization of the elements of \( Y_p \) whose orbits are finite. Then prove that the set of elements of \( Y_p \) whose orbits are finite is dense in \( Y_p \).

f) Let \( W \) be the subset of \( Y_2 \) consisting of elements for which a 2 is always followed by a 1, that is, if \( x_k = 2 \) then \( x_{k+1} = 1 \). Prove that \( W \) is a closed subset of \( Y_2 \). Show that \( W \) is carried into itself by \( T \) and \( T^{-1} \), and so by the action of \( \mathbb{Z} \) defined in part c). (Then \( W \) with this action is an example of a “subshift of finite type”. There are important classes of dynamical systems that are usefully modeled by various subshifts of finite type. This topic is called “symbolic dynamics”. I strongly encourage you to glance at the Wikipedia entries for “symbolic dynamics” and “subshift of finite type”.)

3. A function \( f \) from one topological space to another is said to be open if \( f(O) \) is open for every open set \( O \). The function \( f \) is said to be closed if \( f(C) \) is closed for every closed set \( C \).

Let \( X = [0, 1] \times \mathbb{R} \), with the product topology (which is the same as its relative topology as a subset of \( \mathbb{R}^2 \)). Let \( \pi \) be the usual projection from \( X \) onto \( [0, 1] \).

Let \( S \) be the collection of all subsets, \( A \), of \( X \) such that \( \pi(A) = [0, 1] \), and let \( \pi_A \) denote the restriction of \( \pi \) to \( A \). Give each \( A \in S \) its relative topology from \( X \). Find elements of \( S \) such that:

a) \( \pi_A \) is both open and closed, but is not one-to-one.

b) \( \pi_A \) is not open but is closed.

c) \( \pi_A \) is open but not closed.

d) \( \pi_A \) is not open and not closed.

For each of your examples determine the quotient topology on \( [0, 1] \) coming from viewing \( [0, 1] \) as a quotient of \( A \) via \( \pi_A \).