Math 55: Discrete Math
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Section	*Problems*
Lenstra	6,7,8,9,10

#6. Four boxes contain red balls, white balls and blue balls:
the first box contains 13 red balls, 11 white balls and 6 blue balls;
the second box contains 13 blue balls, 11 red balls and 6 white balls;
the third box contains 13 white balls, 11 blue balls and 6 red balls;
the fourth box contains 10 red balls, 10 white balls and 10 blue balls.
One performs the following experiment: first a box is drawn at random (giving equal probability to all four boxes). Next 12 balls are drawn at random from this box.
(a) Prove that the expected number of red balls that are drawn is equal to the expected number of white balls that are drawn.
(b) What is the expected number of blue balls that are drawn? *Hint:* **avoid doing any calculations.**

Let R_n be the event "exactly n red balls are drawn", W_n the event "exactly n white balls are drawn" and C_j the event "the jth box is chosen". The expected number of red balls drawn is

$$\sum_{n=0}^{12} nP[R_n] = \sum_{j=1}^{4}\sum_{n=0}^{12} nP[R_n \cap C_j] = \sum_{j=1}^{4}\sum_{n=0}^{12} nP[R_n|B_j]P[C_j].$$

By symmetry, for each n one has $P[R_n|C_1] = P[W_n|C_3]$, $P[R_n|B_2] = P[W_n|C_1]$, $P[R_n|B_3] = P[W_n|C_2]$ and $P[R_n|B_4] = P[W_n|C_4]$. Since $P[C_j] = \frac{1}{4}$ is the same for each j, this implies that

$$\sum_{j=1}^{4}\sum_{n=0}^{12} nP[R_n|C_j]P[C_j] = \sum_{j=1}^{4}\sum_{n=0}^{12} nP[W_n|C_j]P[C_j]$$

i.e., the expected number of red balls drawn is the same as the expected number of white balls drawn. The same argument shows that the expected number of blue balls drawn is equal to the expected number of red balls drawn. Now, 12 = (expected number of balls drawn) = (expected number of red balls drawn) + (expected number of white balls drawn) + (expected number of blue balls drawn) = 3 $\times$ (expected number of white balls drawn), and it follows that the expected number of balls drawn of each color is 4.

#7. Let $S = \{1, 2, 3, \ldots, 899, 900\}$, and let each element of S have probability $\frac{1}{900}$. Define the random variables $f, g : S \to R$ by
$f(x)$ = remainder of x upon division by 3,
$g(x)$ = remainder of x upon division by 4.
(a) Prove that f and g are independent.
(b) Calculate $E(f+g)$, $E(fg)$ and $V(f+g)$.

Fix $0 \le a \le 2$ and $0 \le b \le 3$. We must verify that

$$P(f = a, g = b) = P(f = a)P(g = b).$$

There are exactly 300 solutions of $x \equiv a \pmod 3$ in S and 225 solutions to $x \equiv b \pmod 4$ in S, so $P(f = a)P(g = b) = \frac{300}{900}\frac{225}{900} = \frac{1}{12}$. The Chinese remainder theorem says $x \equiv a \pmod 3$, $x \equiv b \pmod 4$ has a unique solution y with $1 \le y \le 12$, and every solution looks like $x = y + 12n$; such a solution is in S if and only if $0 \le n \le 74$, so there are exactly 75 solutions is S. Thus, $P(f = a, g = b) = \frac{75}{900} = \frac{1}{12}$ as required.

$E(f+g) = E(f) + E(g) = \frac{1}{3}(0+1+2) + \frac{1}{4}(0+1+2+3) = \frac{5}{2}$.
$E(fg) = E(f)E(g) = 1 \cdot \frac{3}{2} = \frac{3}{2}$.
$V(f+g) = V(f) + V(g) = E(f^2) - (Ef)^2 + E(g^2) - (Eg)^2 = \frac{1}{3}(1^2+2^2) - 1 + \frac{1}{4}(1^2+2^2+3^2) - \frac{9}{4} = \frac{8}{3}$.

#8. For this exercise, you have to know that Martians come in three sexes: male, emale and female.

A Martian triple is going to have 9 children. At each birth, the probability for each sex is $\frac{1}{3}$. What is the expected number of emales among the children? What is the variance?

Let $f_i = 1$ if the ith child is an emale and $f_i = 0$ otherwise. Then $f_1, f_2, \ldots, f_9$ are independent random variables and the number of emales is $\sum_{i=1}^{9} f_i$. The expected number of emales is $E(\sum_{i=1}^{9} f_i) = \sum_{i=1}^{9} E(f_i) = 9E(f_1) = 9 \cdot \frac{1}{3} = 3$, and the variance is $V(\sum_{i=1}^{9} f_i) = \sum_{i=1}^{9} V(f_i) = 9V(f_1) = 9 \cdot \frac{2}{9} = 2$.

#9. The *Collected Novels* of Mrs. Kip Vandegril have been published in ten volumes. Someone checks out five of these volumes from the local library, chosen at random, without knowing that Mrs. Vandegril wrote only five novels, each occupying two volumes. Calculate, for each $i = 0, 1, 2$, the probability that he can read i complete novels. What is the expected number of complete novels he can read? And the variance?

There are $\binom{10}{5} = 252$ ways of checking out five volumes. In order to check out five volumes without getting both volumes of any novel, one would have to get either the first or the second volume of each of the five novels (and not both); there are $2^5 = 32$ ways of doing this. Thus, the probability of getting 0 complete novels is $\frac{32}{252} = \frac{8}{63}$.

There are $\binom{5}{2} = 10$ ways of selecting both volumes of each of two novels, and 6 ways of choosing 1 of the remaining six volumes, so the probability of getting 2 complete novels is $\frac{10 \cdot 6}{252} = \frac{5}{21}$.

The probability of getting exactly one complete novel must be $1 - \frac{8}{63} - \frac{5}{21} = \frac{40}{63}$.

The expected number of novels our uninformed reader can read is $1 \cdot \frac{40}{63} + 2 \cdot \frac{5}{21} = \frac{10}{9}$ and the variance is $1 \cdot \frac{40}{63} + 4 \cdot \frac{5}{21} - \left(\frac{10}{9}\right)^2 = \frac{200}{567}$.

#10. Let p be a real number, $0 < p < 1$, and $q = 1 - p$. Consider, as in Example 8, the sample space $S = \{h, mh, mmh, mmmh, \ldots\}$, where the nth element has probability $q^{n-1}p$. Let further the random variable $f : S \to R$ map the nth variable to n. In Example 8 we saw that $E(f) = 1/p$. Prove that $V(f) = q/p^2$.

$V(f) = E(f^2) - (Ef)^2 = \sum_{n=1}^{\infty} n^2 q^{n-1} p - 1/p^2 = pq \cdot \frac{q+1}{q(1-q)^3} - 1/p^2 = \frac{q+1}{p^2} - 1/p^2 = q/p^2$.