Math 55: Discrete Math

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Homework Assignment #10

Section	*Problems*
§4.6	2, 6, 8, 9, 16, 21
§5.5	2, 8, 10, 11, 14, 15

(§4.6, #2) In how many different ways can five elements be selected in order from a set with five elements when repetition is allowed?

Since order matters, and repetition is allowed, the answer is

$$5^5 = 3,125.$$

(§4.6, # 6) How many ways are there to select five unordered elements from a set with three elements when repetition is allowed?

Since order does not matter, and repetition is allowed, the answer is

$$\frac{(5+3-1)!}{5!(3-1)!} = \binom{7}{5} = 21.$$

(§4.6, #8) How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?

Since order does not matter, and repetition is allowed, the answer is

$$\frac{(12+21-1)!}{12!(21-1)!} = \binom{32}{12} = 225,792,840.$$

(§4.6, #9) A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose

a) six bagels?

There are eight varieties of bagels. Order does not matter, and repetition is allowed, so the answer is

$$\frac{(6+8-1)!}{6!(8-1)!} = \binom{13}{6} = 1,716.$$

b) a dozen bagels?

Order does not matter and repetition is allowed, so the answer is

$$\frac{(12+8-1)!}{12!(8-1)!} = \binom{19}{12} = 50,388.$$

c) two dozen bagels?

Order does not matter and repetition is allowed, so the answer is

$$\frac{(24+8-1)!}{24!(8-1)!} = \binom{31}{24} = 2,629,575.$$

d) a dozen bagels with at least one of each kind?

Here, after we put one kind of each bagel in our bag, we must still choose four more bagels from the eight varieties. Therefore the answer is

$$\frac{(4+8-1)!}{4!(8-1)!} = \binom{11}{4} = 330.$$

e) a dozen bagels with at least three egg bagels and no more than two salty bagels?

First we put three egg bagels in the bag. At this point the number of ways to pick the remaining 9 bagels is

$$\frac{(9+8-1)!}{9!(8-1)!} = \binom{16}{9} = 11,440.$$

But how many of these include a pick of more than two salty bagels? If three salty bagels are picked (in addition to the three egg bagels), then the number of ways to choose the remaining 6 bagels is

$$\frac{(6+8-1)!}{6!(8-1)!} = \binom{13}{6} = 1,716.$$

So the answer is

$$11,440 - 1,716 = 9,724.$$

(§4.6, #16) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$$

where x_i, $i = 1, 2, 3, 4, 5, 6$ is a nonnegative integer such that

a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?

We will treat these x_i as types of bagels to put in our bag of 29. Since each x_i must be at least 2, we begin by choosing two of each bagel. Then we have to choose 17 more from the 6 types. Order does not matter, and repetition is allowed, so the answer is

$$\frac{(17+6-1)!}{17!(6-1)!} = \binom{22}{17} = 26,334.$$

b) $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 > 5$, $x_6 \geq 6$?

After putting the requisite 22 bagels in the bag, we must still choose seven. Therefore the answer is

$$\frac{(7+6-1)!}{7!(6-1)!} = \binom{12}{7} = 792.$$

(Also, if you misread the problem (it is unclear to me whether or not this was a deliberate curve ball by the author or a typo) as "$x_5 \geq 5$," your answer should be 1,287.)

c) $x_1 \leq 5$?

The total number of ways for choosing the 29 from the 6 types is

$$\frac{(29+6-1)!}{29!(6-1)!} = \binom{34}{29} = 278,256.$$

Of these, how many have $x_1 > 5$? If we first pick 6 x_1 bagels, then we still must pick 23 bagels from the 6 types. The number of ways to do this is

$$\frac{(23+6-1)!}{23!(6-1)!} = \binom{28}{23} = 98,280.$$

So the answer is

$$278,256 - 98,280 = 179,976.$$

d) $x_1 < 8$ and $x_2 > 8$?

First, we pick the requisite 9 of x_2. Then we still must pick 20 more from the six types. The number of ways to do this is

$$\frac{(20+6-1)!}{20!(6-1)!} = \binom{25}{20} = 53,130.$$

But for how many of these is $x_1 \geq 8$? If we pick 8 of x_1 (in addition to the already picked 9 of x_2) there are 12 remaining to be picked from these six types. The number of ways to do this is

$$\frac{(12+6-1)!}{12!(6-1)!} = \binom{17}{12} = 6,188.$$

So the answer is

$$53,130 - 6,188 = 46,942.$$

(§4.6, #21) How many positive integers less than $1,000,000$ have the sum of their digits equal to 19?

How many ways are there of writing six digits whose sum is 19, but each of which is (obviously) less than 10? First, let's imagine that the digits can be anything (in other words, forget the less-than-10 requirement). The number of such six-digit numbers is then

$$\frac{(19+6-1)!}{19!(6-1)!} = \binom{24}{19} = 42,504.$$

But now how many of these are invalid, that is include a digit greater than 9? This problem might get very complicated, except that since the sum of these digits is 19, we know that at most one of these digits is too big. (If two were, the sum would be at least 20.) Imagine the first digit is. So if we place 10 in the first digit, the number of ways of distributing the remaining 9 between the six digits is

$$\frac{(9+6-1)!}{9!(6-1)!} = \binom{14}{9} = 2,002.$$

Since there are $2,002$ possible ways for the first digit to be too big, there are also $2,002$ possible ways for the second and third, and so on, to be too big. So the total number of bad combinations is

$$6(2,002) = 12,012.$$

Therefore, the total number of good combinations is

$$42,504 - 12,012 = 30,492.$$

In other words, there are $30,492$ numbers less than a million whose digits sum to 19.

(§5.5, #2) Of 1000 applicants for a mountain-climbing trip in the Himalayas, 450 get altitude sickness, 622 are not in good enough shape, and 30 have allergies. An applicant qualifies if and only if this applicant does not get altitude sickness, is in good shape, and does not have allergies. If there are 111 applicants who get altitude sickness and are not in good enough shape, 14 who get altitude sickness and have allergies, 18 who are not in good enough shape and have allergies, and 9 who get altitude sickness, are not in good enough shape, and have allergies, how many applicants qualify?

We use standard inclusion-exclusion principles:

$$1000 - (450 + 622 + 30) + (111 + 14 + 18) - (9) = 32,$$

so 32 applicants qualify.

(§5.5, #8) How many onto functions are there from a set with seven elements to one with five elements?

Here we use the onto-functions formula:

$$\begin{aligned} & 5^7 - \binom{5}{4}4^7 + \binom{5}{3}3^7 - \binom{5}{2}2^7 + \binom{5}{1}1^7 \\ = \; & 78,125 - 81,920 + 21,870 - 1,280 + 5 \\ = \; & 16,800 \end{aligned}$$

So there are $16,800$ onto functions from a set with seven elements to one with five.

(§5.5, #10) In how many ways can eight distinct balls be distributed into three distinct urns if each urn must contain at least one ball?

This question asks the number of onto function from a set with eight elements (balls) to a set with three elements (urns). So we solve using the onto-functions formula:

$$3^8 - \binom{3}{2}2^8 + \binom{3}{1}1^8 = 6,561 - 768 + 3 = 5,796$$

So there are $5,796$ such ways.

(§5.5, #11) In how many ways can seven different jobs be assigned to four different employees so that each employee is assigned to at least one job and the most difficult job is assigned to the best employee?

First, we use the onto-functions formula to find the total number of ways of dividing the jobs so that everyone has a job (without additional requirements):

$$\begin{aligned} & 4^7 - \binom{4}{3}3^7 + \binom{4}{2}2^7 - \binom{4}{1}1^7 \\ = \;& 16,384 - 8,748 + 768 - 4 \\ = \;& 8,400 \end{aligned}$$

Then exactly $1/4$ of these have assigned the most difficult job to the best employee. So the number of ways to assign jobs such that everyone has a job *and* the most difficult job is given to the best employee is $2,100$.

(§5.5, #14) What is the probability that none of 10 people receives the correct hat if a hatcheck person hands their hats back randomly?

The answer is the number of derangements of ten objects divided by the total number of permutations of ten objects, which is $D_{10}/10!$.

(§5.5, #15) A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is the probability that in a group of 100 letters

a) no letter is put into the correct envelope?

The answer is the number of derangements of 100 objects over the number of permutations of 100 objects, which is $D_{100}/100!$.

b) exactly 1 letter is put into the correct envelope?

There are 100 choices for the matching envelope-letter pair. For each of these choices, a derangement of the other 99 letters will produce a situation where exactly 1 letter is put into the correct envelope. Therefore, the number of possible ways to do this is $100 \cdot D_{99}$, and therefore the probability that this will happen is $100 \cdot D_{99}/100! = D_{99}/99!$.

c) exactly 98 letters are put into the correct envelope?

Such a permutation would consist merely of a *transposition*, that is, switching two letters before putting the letters in their envelopes. So the number of ways to do this is the number of pairs of letters, which is $\binom{100}{2}$. So the probability that this will happen is

$$\binom{100}{2}/100!$$

d) exactly 99 letters are put into the correct envelope?

After we choose which 99 letters get put into the correct envelopes, we must derange the last letter. One must be deranged to derange one object, so the answer is 0.

e) all letters are put into the correct envelope?

There is one such permutation, and the odds of hitting it by random are $1/10!$.