Math 55: Discrete Math

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Homework Assignments #1 & 2:

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(§1.1, #1) Which of the following sentences are propositions? What are the truth values of those propositions?

a) Boston is the capital of Massachusetts.
True proposition

b) Miami is the capital of Florida.
False proposition (Tallahassee is the capital of Florida.)

c) $2 + 3 = 5$.
True proposition

d) $5 + 7 = 10$.
False proposition

e) $x + 2 = 11$.
Not a proposition

f) Answer this question.
Not a proposition

g) $x + y = y + x$ for every pair of real numbers x and y.
True proposition

(§1.1, #4) Let p and q be the propositions

p: I bought a lottery ticket this week.

q: I won the million dollar jackpot on Friday.

Express each of the following propositions as an English sentence.

a) $\neg p$
"I did not buy a lottery ticket this week."

b) $p \vee q$
"I either bought a lottery ticket this week or won the million dollar jackpot on Friday (or both)."

c) $p \to q$
"If I bought a lottery ticket this week, then I won the million dollar jackpot on Friday."

d) $p \wedge q$
"I bought a lottery ticket this week and won the million dollar jackpot on Friday."

e) $p \leftrightarrow q$
"I won the million dollar jackpot this week if and only if I bought a lottery ticket this week."

f) $\neg p \to \neg q$
"If I did not buy a lottery ticket this week, then I didn't win the million dollar jackpot on Friday."

g) $\neg p \wedge \neg q$

"I neither bought a lottery ticket this week nor won the million dollar jackpot on Friday."

h) $\neg p \vee (p \wedge q)$

"Either I didn't buy a lottery ticket this week, or I both bought a lottery ticket and won the million dollar jackpot on Friday." (Notice the equivalence to sentence **c**.)

(§1.1, #7) Let p and q be the propositions

p: You drive over 65 miles per hour.

q: You get a speeding ticket.

Write the following propositions using p and q and logical connectives.

a) You do not drive over 65 miles per hour.

$\neg p$

b) You drive over 65 miles per hour, but you do not get a speeding ticket.

$p \wedge \neg q$

c) You will get a speeding ticket if you drive over 65 miles per hour.

$q \leftarrow p$

d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.

$\neg p \rightarrow \neg q$

e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.

$p \rightarrow q$

f) You get a speeding ticket, but you do not drive over 65 miles per hour.

$q \wedge \neg p$

g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

$q \rightarrow p$

(§1.2, #9) Show that each implication is a tautology without using truth tables.

(A more "strightforward" answer can be found in the back of the book.)

a) $(p \wedge q) \rightarrow p$

$\Leftrightarrow \neg(p \wedge q) \vee p$	by Example 3
$\Leftrightarrow (\neg p \vee \neg q) \vee p$	by the first De Morgan's law
$\Leftrightarrow p \vee (\neg p \vee \neg q)$	by the first commutative law
$\Leftrightarrow (p \vee \neg p) \vee \neg q$	by the first associative law
$\Leftrightarrow \mathbf{T} \vee \neg q$	by Example 1
$\Leftrightarrow \neg q \vee \mathbf{T}$	by the first commutative law
$\Leftrightarrow \mathbf{T}$	by the first domination law

b) $p \rightarrow (p \vee q)$

$\Leftrightarrow \neg p \vee (p \vee q)$	by Example 3
$\Leftrightarrow (\neg p \vee p) \vee q$	by the first associative law
$\Leftrightarrow (p \vee \neg p) \vee q$	by the first commutative law
$\Leftrightarrow \mathbf{T} \vee q$	by Example 1
$\Leftrightarrow q \vee \mathbf{T}$	by the first commutative law
$\Leftrightarrow \mathbf{T}$	by the first domination law

c) $\neg p \rightarrow (p \rightarrow q)$

$\Leftrightarrow \neg(\neg p) \vee (p \rightarrow q)$	by Example 3
$\Leftrightarrow p \vee (p \rightarrow q)$	by the double negation law
$\Leftrightarrow p \vee (\neg p \vee q)$	by Example 3
$\Leftrightarrow (p \vee \neg p) \vee q$	by the first associative law
$\Leftrightarrow \mathbf{T} \vee q$	by Example 1
$\Leftrightarrow q \vee \mathbf{T}$	by the first commutative law
$\Leftrightarrow \mathbf{T}$	by the first domination law

d) $(p \wedge q) \rightarrow (p \rightarrow q)$

$\Leftrightarrow \neg(p \wedge q) \vee (p \rightarrow q)$	by Example 3
$\Leftrightarrow (\neg p \vee \neg q) \vee (p \rightarrow q)$	by the first De Morgan's law
$\Leftrightarrow (\neg p \vee \neg q) \vee (\neg p \vee q)$	by Example 3
$\Leftrightarrow [(\neg p \vee \neg q) \vee \neg p] \vee q$	by the first associative law
$\Leftrightarrow [\neg p \vee (\neg p \vee \neg q)] \vee q$	by the first commutative law
$\Leftrightarrow \neg p \vee [(\neg p \vee \neg q) \vee q]$	by the first associative law
$\Leftrightarrow \neg p \vee [\neg p \vee (\neg q \vee q)]$	by the first associative law
$\Leftrightarrow \neg p \vee [\neg p \vee (q \vee \neg q)]$	by the first commutative law
$\Leftrightarrow \neg p \vee (\neg p \vee \mathbf{T})$	by Example 1
$\Leftrightarrow \neg p \vee \mathbf{T}$	by the first domination law
$\Leftrightarrow \mathbf{T}$	by the first domination law

e) $\neg(p \rightarrow q) \rightarrow p$

$\Leftrightarrow \neg(\neg p \vee q) \rightarrow p$	by Example 3
$\Leftrightarrow \neg[\neg(\neg p \vee q)] \vee p$	by Example 3
$\Leftrightarrow (\neg p \vee q) \vee p$	by the double negation law
$\Leftrightarrow (q \vee \neg p) \vee p$	by the first commutative law
$\Leftrightarrow q \vee (\neg p \vee p)$	by the first associative law
$\Leftrightarrow q \vee (p \vee \neg p)$	by the first commutative law
$\Leftrightarrow q \vee \mathbf{T}$	by Example 1
$\Leftrightarrow \mathbf{T}$	by the first domination law

f) $\neg(p \rightarrow q) \rightarrow \neg q$

$\Leftrightarrow \neg(\neg p \vee q) \rightarrow \neg q$	by Example 3
$\Leftrightarrow \neg[\neg(\neg p \vee q)] \vee \neg q$	by Example 3
$\Leftrightarrow (\neg p \vee q) \vee \neg q$	by the double negation law
$\Leftrightarrow \neg p \vee (q \vee \neg q)$	by the first associative law
$\Leftrightarrow \neg p \vee \mathbf{T}$	by Example 1
$\Leftrightarrow \mathbf{T}$	by the first domination law

(§1.2, #10) Show that each implication is a tautology without using truth tables.

a) $[\neg p \wedge (p \vee q)] \rightarrow q$

$\Leftrightarrow [(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$	by the second distributive law
$\Leftrightarrow [(p \wedge \neg p) \vee (\neg p \wedge q)] \rightarrow q$	by the first commutative law
$\Leftrightarrow [\mathbf{F} \vee (\neg p \wedge q)] \rightarrow q$	by Example 1
$\Leftrightarrow [(\neg p \wedge q) \vee \mathbf{F}] \rightarrow q$	by the first commutative law
$\Leftrightarrow (\neg p \wedge q) \rightarrow q$	by the second identity law
$\Leftrightarrow \neg(\neg p \wedge q) \vee q$	by Example 3
$\Leftrightarrow [\neg(\neg p) \vee \neg q] \vee q$	by the first De Morgan's law
$\Leftrightarrow \neg(\neg p) \vee (\neg q \vee q)$	by the first associative law
$\Leftrightarrow \neg(\neg p) \vee (q \vee \neg q)$	by the first commutative law
$\Leftrightarrow \neg(\neg p) \vee \mathbf{T}$	by Example 1
$\Leftrightarrow \mathbf{T}$	by the first domination law

b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

$\Leftrightarrow \neg[(\neg p \vee q) \wedge (\neg q \vee r)] \vee (\neg p \vee r)$	by Example 3 (4 $\times$)
$\Leftrightarrow [\neg(\neg p \vee q) \vee \neg(\neg q \vee r)] \vee (\neg p \vee r)$	by the first De Morgan's law
$\Leftrightarrow [(\neg[\neg p] \wedge \neg q) \vee (\neg[\neg q] \wedge \neg r)] \vee (\neg p \vee r)$	by the second De Morgan's law
$\Leftrightarrow [(p \wedge \neg q) \vee (q \wedge \neg r)] \vee (\neg p \vee \neg[\neg r])$	by the double negation law (3 $\times$)
$\Leftrightarrow ([(p \wedge \neg q) \vee q] \wedge [(p \wedge \neg q) \vee \neg r]) \vee (\neg p \vee \neg[\neg r])$	by the first distributive law
$\Leftrightarrow ([q \vee (p \wedge \neg q)] \wedge [\neg r \vee (p \wedge \neg q)]) \vee (\neg[\neg r] \vee \neg p)$	by the first commutative law (2 $\times$)
$\Leftrightarrow [([q \vee p] \wedge [q \vee \neg q]) \wedge (\neg r \vee [p \wedge \neg q])] \vee (\neg[\neg r] \vee \neg p)$	by the first distributive law
$\Leftrightarrow [([q \vee p] \vee \mathbf{T}) \wedge (\neg r \vee [p \wedge \neg q])] \vee (\neg[\neg r] \vee \neg p)$	by Example 1
$\Leftrightarrow [(q \vee p) \wedge (\neg r \vee [p \wedge \neg q])] \vee (\neg[\neg r] \vee \neg p)$	by the first identity law
$\Leftrightarrow [([q \vee p] \wedge \neg r) \vee ([q \vee p] \wedge p \wedge \neg q)] \vee (\neg[\neg r] \vee \neg p)$	by the second distributive law
$\Leftrightarrow [(\neg r \wedge [q \vee p]) \vee ([q \vee p] \wedge p \wedge \neg q)] \vee (\neg[\neg r] \vee \neg p)$	by the second commutative law
$\Leftrightarrow [(\neg r \wedge q) \vee (\neg r \wedge p) \vee ([q \vee p] \wedge p \wedge \neg q)] \vee (\neg[\neg r] \vee \neg p)$	by the second distributive law
$\Leftrightarrow [(\neg r \wedge q) \vee ([q \vee p] \wedge p \wedge \neg q) \vee (\neg r \wedge p)] \vee (\neg[\neg r] \vee \neg p)$	by the first commutative law
$\Leftrightarrow [(\neg r \wedge q) \vee ([q \vee p] \wedge p \wedge \neg q)] \vee [(\neg r \wedge p) \vee (\neg[\neg r] \vee \neg p)]$	by the first associative law
$\Leftrightarrow [(\neg r \wedge q) \vee ([q \vee p] \wedge p \wedge \neg q)] \vee [(\neg r \wedge p) \vee \neg(\neg r \wedge p)]$	by the first De Morgan's law
$\Leftrightarrow [(\neg r \wedge q) \vee ([q \vee p] \wedge p \wedge \neg q)] \vee \mathbf{T}$	by Example 1
$\Leftrightarrow \mathbf{T}$	by the first domination law

c) $[p \wedge (p \rightarrow q)] \rightarrow q$

$\Leftrightarrow \neg[p \wedge (p \rightarrow q)] \vee q$	by Example 3
$\Leftrightarrow [\neg p \vee \neg(p \rightarrow q)] \vee q$	by the first De Morgan's law
$\Leftrightarrow \neg p \vee [\neg(p \rightarrow q) \vee q]$	by the first associative law
$\Leftrightarrow \neg p \vee [q \vee \neg(p \rightarrow q)]$	by the first commutative law
$\Leftrightarrow (\neg p \vee q) \vee \neg(p \rightarrow q)$	by the first associative law
$\Leftrightarrow (p \rightarrow q) \vee \neg(p \rightarrow q)$	by Example 3
$\Leftrightarrow \mathbf{T}$	by Example 1

d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

$\Leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r) \wedge (p \vee q)] \rightarrow r$	by the second commutative law
$\Leftrightarrow [([p \rightarrow r] \wedge [q \rightarrow r] \wedge p) \vee ([p \rightarrow r] \wedge [q \rightarrow r] \wedge q)] \rightarrow r$	by the second distributive law
$\Leftrightarrow [([q \rightarrow r] \wedge p \wedge [p \rightarrow r]) \vee ([p \rightarrow r] \wedge q \wedge [q \rightarrow r])] \rightarrow r$	by the second commutative law (2 $\times$)
$\Leftrightarrow [([q \rightarrow r] \wedge [p \wedge (\neg p \vee r)]) vee ([p \rightarrow r] \wedge [q \wedge (\neg q \vee r)]) \rightarrow r$	by Example 3
$\Leftrightarrow [([q \rightarrow r] \wedge [(p \wedge \neg p) \vee (p \wedge r)]) \vee ([p \rightarrow r] \wedge [(q \wedge \neg q) \vee (q \wedge r)])] \rightarrow r$	by the second distributive law (2 $\times$)
$\Leftrightarrow [([q \rightarrow r] \wedge [\mathbf{F} \vee (p \wedge r)]) \vee ([p \rightarrow r] \wedge [\mathbf{F} \vee (q \wedge r)])] \rightarrow r$	by Example 1 (2 $\times$)
$\Leftrightarrow [([q \rightarrow r] \wedge [(p \wedge r) \vee \mathbf{F}]) \vee ([p \rightarrow r] \wedge [(q \wedge r) \vee \mathbf{F}])] \rightarrow r$	by the first commutative law (2 $\times$)
$\Leftrightarrow [([q \rightarrow r] \wedge p \wedge r) \vee ([p \rightarrow r] \wedge q \wedge r)] \rightarrow r$	by the second identity law (2 $\times$)
$\Leftrightarrow [(r \wedge p \wedge [q \rightarrow r]) \vee (r \wedge q \wedge [p \rightarrow r])] \rightarrow r$	by the second commutative law (2 $\times$)
$\Leftrightarrow [r \wedge ([p \wedge (q \rightarrow r)] \vee [q \wedge (p \rightarrow r)])] \rightarrow r$	by the second distributive law
$\Leftrightarrow \neg[r \wedge ([p \wedge (q \rightarrow r)] \vee [q \wedge (p \rightarrow r)])] \vee r$	by Example 3
$\Leftrightarrow [\neg r \vee \neg([p \wedge (q \rightarrow r)] \vee [q \wedge (p \rightarrow r)])] \vee r$	by the first De Morgan's law
$\Leftrightarrow r \vee [\neg r \vee \neg([p \wedge (q \rightarrow r)] \vee [q \wedge (p \rightarrow r)])]$	by the first commutative law
$\Leftrightarrow (r \vee \neg r) \vee \neg([p \wedge (q \rightarrow r)] \vee [q \wedge (p \rightarrow r)])$	by the first associative law
$\Leftrightarrow \mathbf{T} \vee \neg([p \wedge (q \rightarrow r)] \vee [q \wedge (p \rightarrow r)])$	by Example 1
$\Leftrightarrow \neg([p \wedge (q \rightarrow r)] \vee [q \wedge (p \rightarrow r)[) \vee \mathbf{T}$	by the first commutative law
$\Leftrightarrow \mathbf{T}$	by the first domination law

(§1.2, #11) Verify the following equivalences, which are known as the *absorption laws*.

a) $[p \vee (p \wedge q)] \Leftrightarrow p$

We solve using a truth table:

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	**T**	**T**	**T**
T	**F**	**F**	**T**
F	**T**	**F**	**F**
F	**F**	**F**	**F**

This shows that $[p \vee (p \wedge q)] \Leftrightarrow p$.

b) $[p \wedge (p \vee q)] \Leftrightarrow p$

We solve using a truth table:

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	**T**	**T**	**T**
T	**F**	**T**	**T**
F	**T**	**T**	**F**
F	**F**	**F**	**F**

This shows that $[p \wedge (p \vee q)] \Leftrightarrow p$.

(§1.2, #18) Show that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.

$$\begin{aligned}
\neg(p \oplus q) &\Leftrightarrow \neg[(p \wedge \neg q) \vee (\neg p \wedge q)] && \text{by the definition of } \oplus \\
&\Leftrightarrow \neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan's law} \\
&\Leftrightarrow (\neg p \vee \neg[\neg q]) \wedge (\neg[\neg p] \vee \neg q) && \text{by the first De Morgan's law } (2\times) \\
&\Leftrightarrow (\neg p \vee q) \wedge (p \vee \neg q) && \text{by the double negation law } (2\times) \\
&\Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p) && \text{by the first commutative law} \\
&\Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) && \text{by Example 3 } (2\times) \\
&\Leftrightarrow p \leftrightarrow q && \text{by the definition of } \leftrightarrow
\end{aligned}$$

This shows that $\neg(p \oplus q)$ is logically equivalent to $p \leftrightarrow q$.

(§1.3, #8) Let $Q(x, y)$ be the statement "x has been a contestant on y." Express each of the following sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the universe of discourse for x is the set of all students at your school and for y is the set of all quiz shows on television.

a) There is a student at your school who has been a contestant on a television quiz show.
$\exists x\, \exists y\, Q(x, y)$
b) No student at your school hes ever been a contestant on a television quiz show.
$\forall x\, \forall y\, \neg Q(x, y)$
c) There is a student at your school who has been a contestant on *Jeopardy* and *Wheel of Fortune*.
$\exists x\, (Q(x, \mathit{Jeopardy}) \wedge Q(x, \mathit{Wheel\ of\ Fortune}))$
d) Every television quiz show has had a student from your school as a contestant.
$\forall y\, \exists x\, Q(x, y)$
e) At least two students from your school have been contestants on *Jeopardy*.
$\exists x_1, x_2\, (Q(x_1, \mathit{Jeopardy}) \wedge Q(x_2, \mathit{Jeopardy}) \wedge x_1 \neq x_2)$

(§1.3, #9) Let $L(x, y)$ be the statement "x loves y," where the universe of discourse for both x and y is the set of all the people in the world. Use quantifiers to express each of the following statements:

a) Everybody loves Jerry.
$\forall x\, L(x, \text{Jerry})$
b) Everybody loves somebody.
$\forall x\, \exists y\, L(x, y)$
c) There is somebody whom everybody loves.
$\exists y\, \forall x\, L(x, y)$
d) Nobody loves everybody.
$\forall x\, \exists y\, \neg L(x, y)$
e) There is somebody whom Lydia does not love.
$\exists y\, \neg L(\text{Lydia}, y)$
f) There is somebody whom no one loves.
$\exists y\, \forall x\, \neg L(x, y)$
g) There is exactly one person whom everybody loves.
$\exists y_1\, \forall y_2\, (\forall x\, L(x, y_2) \leftrightarrow y_1 = y_2)$
h) There are exactly two people whom Lynn loves.
$\exists y_1, y_2 [\forall y_3\, (L(\text{Lynn}, y_3) \leftrightarrow [y_1 = y_3 \vee y_2 = y_3]) \wedge y_1 \neq y_2]$
i) Everyone loves himself or herself.
$\forall x\, L(x, x)$
j) There is someone who loves no one besides himself or herself.
$\exists x\, \forall y\, [x \neq y \leftrightarrow \neg L(x, y)]$

(§1.3, #17) Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a professor," "x is ignorant," and "x is vain," respectively. Express each of the following statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$, where the universe of discourse is the set of all people.

a) No professors are ignorant.
$P(x) \rightarrow \neg Q(x)$
b) All ignorant people are vain.
$Q(x) \rightarrow R(x)$
c) No professors are vain.
$P(x) \rightarrow \neg R(x)$
***d) Does (c) follow from (a) and (b)? If not, is there a correct conclusion?**

No, **(c)** does not follow from **(a)** and **(b)**. There could well be a professor who is vain but not ignorant. An example of a correct conclusion would be "Anyone who is either a professor or is not vain (or both) is not ignorant." This is written symbolically as $(P(x) \vee \neg R(x)) \rightarrow \neg Q(x)$.

(§1.4, #3) Determine whether each of the following pairs of sets is equal.

a) $\{1,3,3,3,5,5,5,5,5\}$, $\{5,3,1\}$

Yes, they are equal.

b) $\{\{1\}\}$, $\{1,\{1\}\}$

No, they are not equal. 1 is a member of the second set, but not the first.

c) $\emptyset$, $\{\emptyset\}$

No, they are not equal. The first set has no members, and the second set has 1 member, namely $\emptyset$.

(§1.4, #5) Determine whether each of the following statements is true or false.

a) $x \in \{x\}$

True

b) $\{x\} \subseteq \{x\}$

True

c) $\{x\} \in \{x\}$

False

d) $\{x\} \in \{\{x\}\}$

True

e) $\emptyset \subseteq \{x\}$

True. $\emptyset$ is a subset of *any* set.

f) $\emptyset \in \{x\}$

False.

(§1.4, #6) Use a Venn diagram to illustrate the relationship $A \subseteq B$ and $B \subseteq C$.

A B C

(§1.4, #9) What is the cardinality of each of the following sets?

a) $\{a\}$

1

b) $\{\{a\}\}$

1

c) $\{a, \{a\}\}$

2

d) $\{a, \{a\}, \{a, \{a\}\}\}$

3

(§1.4, #10) What is the cardinality of each of the following sets?

a) $\emptyset$

0

b) $\{\emptyset\}$

1

c) $\{\emptyset, \{\emptyset\}\}$

2

d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

3 (You may be interested in knowing that this problem is a demonstration of how the natural numbers are constructed in set theory. "0" is the empty set, "1" is the set containing 0, "2" is the set containing 0 and 1, etc. Such sets are called *ordinal* sets. Question: Can you prove that if S is an ordinal set, and x is an element of S, that x is therefore a *subset* of S?)

(§1.4, #12) Can you conclude that $A = B$ if A and B are two sets with the same power set?

Yes. Suppose $x \in A$ and $x \notin B$. Then $\{x\} \in \wp(A)$, but $\{x\} \notin \wp(B)$. Since $\wp(A) = \wp(B)$, this is a contradiction, and so every element of A is also in B. By a similar argument, every element of B is also in A, so the sets are the same.

In general, a set can be "constructed" from its power set by taking the union of all elements of the power set. In this way,

$$A = \bigcup_{x \in \wp(A)} x = \bigcup_{x \in \wp(B)} x = B$$

(§1.5, #2) Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of the following sets in terms of A and B.

a) the set of sophomores taking discrete mathematics at your school
$A \cap B$

b) the set of sophomores at your school who are not taking discrete mathematics.
$A - B$

c) the set of students at your school who either are sophomores or are taking discrete mathematics
$A \cup B$

d) the set of students at your school who either are not sophomores or are not taking discrete mathematics
$\bar{A} \cup \bar{B}$

(§1.5, #4) Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$
$\{a, b, c, d, e, f, g, h\}$

b) $A \cap B$
$\{a, b, c, d, e\}$

c) $A - B$
$\emptyset$

d) $B - A$
$\{f, g, h\}$

(§1.5, #16) Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.
We show the equality using a membership table:

A	B	C	$A-B$	$A-C$	$B-C$	$(A-B)-C$	$(A-C)-(B-C)$
1	1	1	0	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	1	0	0	0	0
1	0	0	1	1	0	1	1
0	1	1	0	0	0	0	0
0	1	0	0	0	1	0	0
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0

This shows that $(A - B) - C = (A - C) - (B - C)$.

(§1.5, # 20) Can you conclude that $A = B$ if A, B, and C are sets such that

a) $A \cup C = B \cup C$?
No. Take $A = \emptyset$ and $B = C = \{\text{Ribet}\}$.

b) $A \cap C = B \cap C$?
No. Take $A = \{\text{Ribet}\}$ and $B = C = \emptyset$.. (In both of these problems, the answer would be yes if "for some set C" were replaced with "for *any* set C.")

(§1.6, #3) Determine whether f is a function from the set of all bit strings to the set of integers if

a) $f(S)$ is the position of a 0 bit in S.

No. The relation f is not defined for bit strings of all 1s, and is not determined for all bit strings with more than one 0 bit.

b) $f(S)$ is the number of 1 bits in S.

Yes. This is a function on all (finite) bit strings.

c) $f(S)$ is the smallest integer i such that the ith bit of S is 1 and $f(S) = 0$ when S is the empty string, the string with no bits.

No. Here the problem lies with bit strings made entirely of 0s. What is, for example. $f(0000)$?

(§1.6, # 4) Find the domain and range of the following functions.

a) the function that assigns to each nonnegative integer its last digit

The domain, as stated, is $\mathbb{N}$, the set of nonnegative integers. The range is the set of integers between 0 and 9 (inclusively).

b) the function that assigns the next largest integer to a positive integer

The domain, as stated, is $\mathbb{Z}^+$, the set of positive integers. The range is the set of integers greater than 1.

c) the function that assigns to a bit string the number of 1 bits in the string

The domain is the set of all bit strings. The range is $\mathbb{N}$, the set of nonnegative integers.

d) the function that assigns to a bit string the number of bits in the string

Just as above, the domain is the set of all bit strings and the range is $\mathbb{N}$.

(§1.6, #6) Determine whether each of the following functions from $\{a, b, c, d\}$ to itself is one-to-one.

a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$

Yes, this function is one-to-one.

b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$

No, $f(a) = f(b) = b$.

c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

No, $f(a) = f(d) = d$.

(§1.6, #7) Which functions in Exercise 6 are onto?

The same one. In both **(b)** and **(c)**, the element a is not in the image of f.

(§1.6, #10) Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is

a) one-to-one but not onto.

For example, let $f(x) = 2x$. This is one-to-one, but no odd number is in the image.

b) onto, but not one-to-one.

For example, let $f(x) = \lfloor \frac{x}{2} \rfloor$. Since for any y, $f(2y) = y$, this is onto. But it certainly isn't one-to-one, since $f(0) = f(1) = 0$.

c) both onto and one-to-one (but different from the identity function).

For example, let $f(x) = x + 1$ for all even x, and $f(x) = x - 1$ for all odd x. This switches numbers by pairs, beginning with $\{0, 1\}$, $\{2, 3\}$ and so forth. It is both onto and one-to-one.

d) neither one-to-one nor onto.

For example, let $f(x) = 0$. (Is this really a function?!?) This is neither one-to-one, nor onto.

(§1.6, #15) Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

Suppose there are elements a_1 and a_2 in A such that $(f \circ g)(a_1) = (f \circ g)(a_2)$. We must show, then, that $a_1 = a_2$. Since $(f \circ g)(a_1) = (f \circ g)(a_2)$, this means that $f(g(a_1)) = f(g(a_2))$. So because f is one-to-one, we can infer that $g(a_1) = g(a_2)$. And since g is also one-to-one, we know that $a_1 = a_2$. This proves that $(f \circ g)$ is one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Let c be any element of C. We must show that there is some a in A such that $(f \circ g)(a) = c$. Since f is onto, there is some b in B such that $f(b) = c$. Furthermore, since g is onto, there is some a in A such that $g(a) = b$. Therefore we see that $(f \circ g)(a) = f(g(a)) = f(b) = c$. This proves that $(f \circ g)$ is onto.

(§1.6, #*17) If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

No. The easiest way to see this is by showing a counterexample. Let $A = \{w\}$, let $B = \{x, y\}$ and let $C = \{z\}$. Furthermore, we define the functions $g : A \to B$ and $f : B \to C$ by $g(w) = x$, $f(x) = z$, and $f(y) = z$. From this example, we see that f is certainly onto, and that $f \circ g$ is also onto, but that g is not.

What can be said is that if f is *one-to-one*, and $f \circ g$ is onto, then both f and g must also be onto. Why?