Math 55: Discrete Math
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December 15, 1997
Review Sheet Solutions

1) : For a), the question "Are you a liar?" does not work as both liars and truthtellers tell you "No." For b), it is implicit but not stated that asking a question with known answer such as "Is 2+2=4?" is not allowed. The question begs the answer, "If I were to ask you, 'Are you a liar?', what would you say?" The truthteller says "No", and the liar says "Yes".

2) :

$$a)\forall x F(x, Larry)$$

$$b)\forall y F(Carmen, y)$$

$$c)\forall x \exists y F(x, y)$$

$$d)\neg\exists x \forall y F(x, y)$$

$$e)\forall y \exists x F(x, y)$$

$$f)\neg\exists x(F(x, Barry) \wedge F(x, Jerry))$$

$$g)\exists y_1 \exists y_2 (y_1 \neq y_2) \wedge F(Emmitt, y_1) \wedge F(Emmitt, y_2) \wedge \forall y(F(Emmitt, y) \to (y = y_1 \vee y = y_2))$$

$$h)\exists y_1 \forall x F(x, y_1) \wedge \forall y(\forall x F(x, y) \to y = y_1)$$

$$i)\forall x \neg F(x, x)$$

$$j)\exists x \exists y_1 (x \neq y_1) \wedge F(x, y_1) \wedge \forall y(F(x, y) \to y = y_1)$$

Of course there are alternate ways of saying the same thing for all these.

3) : $\{\{c\}\}$ is one element. The answer is 5.

4) : a) says that $B \subseteq A$. b) says that $A \cap B = \emptyset$. c) says that $A = B$. d) says absolutely nothing. This is a set-theoretic identity.

5) : For a), $x \in A \cup B$ means $x \in A$ or $x \in B$. The former statement implies that $f(x) \in f(A)$, and the latter implies $f(x) \in f(B)$. So $f(A \cup B) \subseteq f(A) \cup f(B)$. Now $f(A) \subseteq f(A \cup B)$, and $f(B) \subseteq f(A \cup B)$, so $f(A) \cup f(B) \subseteq f(A \cup B)$. Thus $f(A \cup B) = f(A) \cup f(B)$. For b), if $x \in A \cap B$, then $x \in A \wedge x \in B$. So $f(x) \in f(A) \wedge f(x) \in f(B)$. So $f(x) \in f(A) \cap f(B)$. So $f(A \cap B) \subseteq f(A) \cap f(B)$. To see that equality does not neccessarily hold, let $A = \{1\}, B = \{2\}, f(1) = f(2) = 1$. So $A \cap B = \emptyset$, so $f(A \cap B) = \emptyset$, but $f(A) \cap f(B) = \{1\}$.

6) : This is easy. The sum is 78.

7) : $x^2 \geq x$, for $x \geq 1$. So if $\exists C \exists K |f(x)| \leq C|x|, x \geq K$, then $|f(x)| \leq C|x^2|, x \geq Max(K, 1)$. So $f(x)$ is $O(x^2)$.

8) : The function $xlogx$ works nicely.

9) : You know how to do this one.

10) : You know how to do this one, too. Just plug right into the Chinese Remainder Theorem.

11) : For a), notice that 18 and 15 are not relatively prime, so the CRT does not apply. They have greatest common factor 3. The first congruence implies that $x \equiv 0 mod 3$, and the second implies that $x \equiv 1 mod 3$. Thus there is no simultaneous solution. For b), The solution 1 works just fine.

12) : You also know how to do this one. Just plug right in.

13) : This one is a bit tricky. The first thing to do is factor $2310 = 2 * 3 * 5 * 7 * 11$. This is not an accident. So 2310^{144} is divisible by all those primes, so if we subtract 1, it's no longer divisible. So our prime is at least 13. Now factor $144 = 12 * 12$. This is not an accident either. So $2310^{144} = (2310^{12})^{12}$, and 2310^{12} is relatively prime to 13. Thus Fermat's Little Theorem says that $2310^{144} \equiv 1 mod 13$. So our number is divisible by 13. So that's the answer.

14) : This is also a tricky one. I completely flailed on this one for quite a while. Here's the trick. There are $2^{10} = 1024$ subsets of S. The possible sums for these subsets range from 1 to $90 + 91 + ... + 99 < 1000$. So Pigeonhole Principle says there are two subsets with the same sum. Now here's the next trick: neither subset may be properly contained in the other, or else the sums would not be equal. Also, the two subsets are different. Let the two subsets be called T_1 and T_2. Then we have said that $\exists t_1 \in T_1 - T_2, \exists t_2 \in T_2 - T_1$. So let's take the two subsets T_1 and T_2 and erase their intersection $T_1 \cap T_2$ from both subsets. The resulting subsets still have the same sum, as we have erased the same thing from both. They're disjoint, as we have erased all of their intersection. But they're both non-empty, as t_1 is in the first resulting set, and t_2 is in the second. So we are done.

15) : There's a misprint here. It should read: Show that $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer for any natural number n. Here's how to do it:

$$n^5 \equiv n mod 5$$

$$n^4 \equiv n mod 2$$

$$n^3 \equiv n mod 3$$

The first and third are from Fermat's Little Theorem, the second one is just an easy property of even and odd numbers. So $n^5/5$ is an integer plus $n/5$,$n^4/2$ is an integer plus $n/2$,$n^3/3$ is an integer plus $n/3$. So our expression is an integer plus $n/5 + n/2 + n/3 - n/30 = n$. So it's an integer.

16) : Let our n numbers be $\{x_1, ...x_n\}$. Define the n numbers:

$$y_1 = x_1$$

$$y_2 = x_1 + x_2$$

$$...$$

$$y_n = x_1 + ... + x_n$$

Now, if any of these are $\equiv 0 mod n$, we are done. So if we're not done, they're all $\not\equiv 0 mod n$, so there are only $n - 1$ possible remainders these n numbers could have $mod n$. So Pigeonhole Principle says that two of the numbers are congruent $mod n$. Let these two numbers be $x_i, x_j, i > j$. Then:

$$y_i - y_j = x_{j+1} + ... + x_i$$

is divisible by n. So we are done.

17) : We first prove the base case:

$$1 = f_1.$$

So now for the inductive step, assume that all numbers $\leq n$ can be written as the sum of distinct Fibonacci numbers. So then look at $n + 1$. The largest Fibonacci number $\leq (n + 1)$ is greater than $(n + 1)/2$. Check this!: if it weren't, the next Fibonacci number would also be $\leq n$... So subtract this Fibonacci number. What remains can be written as the sum of distinct Fibonacci numbers, by inductive hypothesis. Also, from what we've said, these Fibonacci numbers will be distinct from the first one we subtracted. So $n + 1$ can be written as the sum of distinct Fibonacci numbers. By induction, then, all positive integers may be written in such a manner.

18) : To each of the n people at a party, let's assign them a number that counts the number of other people at the party they know. So these n numbers we've assigned will be the pigeons. There are n possible values for this number: a person can know no one there, they can know everyone there, or anywhere in between. So the only way we will not be done is if all n people have a different one of the numbers from 0 to $n - 1$ assigned to them. But this implies that someone knows no one at the party, and someone knows everyone at the party. This is a contradiction. So we are done.

19) : Let's first look at these 55 numbers $\{x_1, ...x_{55}\}$, mod 9. There are 9 possible remainders mod 9, so PP says that there is one remainder that has $Ceiling(55/9) = 7$ of our numbers in it. So now let's pair adjacent numbers less than 101 that differ by 9. That is, if r is the remainder that we have discovered, let's let our first new pigeonhole be the pair $(r, r+9)$, our second one be $(r+2*9, r+3*9)$, ... our sixth be $(r+10*9, r+11*9)$. As our numbers are less than 101, we see that the next multiple would be $r+12*9 = r+108$, which is bigger than 100. So we have only these 6 pigeonholes. But we have 7 pigeons, from before. So PP says that there is a pigeonhole with two pigeons in it. But our numbers are distinct, so these two pigeons in the same pigeonhole must be numbers that differ by 9. We are done.

20) : Base Case:

$$4^{(1+1)} + 5^{(2*1-1)} = 16 + 5 = 21.$$

Now for the inductive step. Assume that:

$$21 | (4^{n+1} + 5^{2n-1}).$$

Then let's compute the $(n+1)^{st}$ term:

$$4^{n+2} + 5^{2n+1} = 4(4^{n+1} + 5^{2n-1}) + (5^{2n+1} - 4 * 5^{2n-1}) = 4(4^{n+1} + 5^{2n-1}) + 21 * 5^{2n-1}.$$

So 21 divides both the first and second terms, so it divides the sum. Thus the $(n+1)^{st}$ term is also divisible by 21. So by induction, we are done.

21) : For a), All we have to do is choose 4 other letters, and then order the whole thing. So there are $\binom{24}{4} * 6!$ such words. For b), let's treat a and b temporarily as one letter *ab*. So we need to choose the other four letters, order the five letters, and then order a and b within the letter *ab*. So there are $\binom{24}{4} * 5! * 2$ such words. For c), we just need to realise that exactly half the answers in part a) have a to the left of b, so the answer is $\binom{24}{4} * 5! * 3$.

22) : For this let's treat the two I's as one letter 'II'. So there are now 12 letters, and there are 2 M's, 2 T's, and 3 A's. So the desired number of words is:

$$\frac{12!}{3!2!2!}.$$

23) : You know exactly how to do this one.

24) : This is straight inclusion-exclusion. The number of numbers divisible by 6 is [1000/6]. The number of numbers divisib le by 10 is [1000/10].The number of numbers divisib le by 14 is [1000/14]. The number of numbers divisible by 6 and 10 is [1000/lcm(6,10)] = [1000/30]. Etc. Etc. We see that the desired answer is:

$$[1000/6] + [1000/10] + [1000/14] - [1000/30] - [1000/42] - [1000/70] + [1000/210].$$

25) : I don't know a quick way to do this. I think we just have to compute the probabilities for 0,2,4,6,8,10, and add them together. Thus the answer is:

$$\frac{D_{10}}{10!} + \frac{D_8 * \binom{10}{2}}{10!} + \frac{D_6 * \binom{10}{4}}{10!} + \frac{D_4 * \binom{10}{6}}{10!} + \frac{D_2 * \binom{10}{8}}{10!} + \frac{1}{10!}.$$

26) : This one's a bit tricky too. The total number of all functions from A to B is 3^{10}. Now the number that completely miss an element of B is $\binom{3}{1} * 2^{10}$. The number of functions that completely miss two elements of B is $\binom{3}{2}$. Now, the number of onto functions that hit an element of B exactly once is $3 * 10 * Onto(9, 2)$: we pick an element of B to hit exactly once, pick an element of A to hit it, and make sure the function is onto the other two elements of B. Now, the number of onto functions that hit two elements of B exactly once is $3 * 10 * 9$: we pick the element of B that gets the other 8, then pick the element of A that gets sent to one of the elements of B, then pick the other. So the total answer is:

$$3^{10} - 3 * 2^{10} + 3 - 3 * 10 * Onto(9, 2) + 3 * 10 * 9.$$

27) : let c be the random variable that tells us the value on the cubic die, and *oct* the r.v. for the octahedral die. So:

$$E(c) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}.$$

$$E(oct) = \frac{0+1+2+3+4+5+6+7}{8} = \frac{7}{2}.$$
$$E(c^2) = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = \frac{91}{6}.$$
$$E(oct^2) = \frac{0^2+...+7^2}{8} = something.$$

So:

$$V(c) = E(c^2) - (E(c))^2 = \frac{35}{12}.$$
$$V(oct) = E(oct^2) - (E(oct))^2 = something.$$

So:

$$E(f) = E(c) + E(oct) = 7.$$

As c and oct are independent:

$$V(f) = V(c) + V(oct) = \frac{35}{12} + something.$$

28) : a): Let R be the event that the block is red, B that it's blue, C that it's a circle, and S that it's a square. So the problem tells us that:

$$P(R)P(S) > P(B)P(C) \geq P(B \cap C) > P(R \cap S).$$

So:

$$P(R \cap S) \neq P(R)P(S).$$

That is, R and S are not independent. For b): a little playing around with possibilities will convince you that the only solution is: two red triangles, two green squares, and a blue circle.

29) : All we need to do is count the number of palindromes less than $10,000$. All one-digit numbers are palindromes, there are 9 of them (no zero). Now, once we pick a digit, a two-digit palindrome has to be two copies of this digit, so there are 9 two-digit palindromes as well (again no zero). Now, once we pick a non-zero digit for the first digit of a 3-digit palindrome, the last digit has to be this as well. But any of the ten digits may be the middle digit- (zero *is* allowed in the interior of a number.) So there are $9*10$ three-digit palindromes. Once we pick the first non-zero digit of a four-digit palindrome, the last digit has to be this as well. Then we may pick any digit for the second, and thus the third digit, as well. So there are also $9*10$ four-digit palindromes. So there is a total of $9+9+90+90 = 198$ palindromes less than $10,000$. So the prob. of picking one is $\frac{198}{10000}$.

30): For a), I don't think this is too important to know, but the generating function for the sum of random variables is the product of the generating functions. So the generating function of f_3 is:

$$g_3 = \frac{z+2}{3}\frac{3}{4-z} = \frac{z+2}{4-z}.$$

For b), remember that the expected value of a random variable is the derivative of its gen. fncn. evaluated at 1. So:

$$g_1'(z) = 1/3.g_1'(1) = 1/3.$$
$$g_2'(z) = 3*(-1)*(4-z)^{-2}*(-1) = 3*(4-z)^{-2}.g_2'(1) = 3/9 = 1/3.$$
$$g_3'(z) = \frac{(4-z)*1-(z+2)*(-1)}{(4-z)^2} = \frac{6}{(4-z)^2}.g_3'(1) = 6/9 = 2/3.$$

For c), remember that the variance of a random variable f with gen. fncn. g is $g''(1)+g'(1)-(g'(1))^2$. Just plug right in. For d), just read off the coefficients of the powers of z: you have to know what $\frac{1}{4-z}$ is for the last two:

$$P(f_1 = 1) = 1/3, P(f_1 = 2) = 2/3, P(f_1 = n) = 0, n > 2.$$

$$\frac{3}{4-z} = 3/4 * \frac{1}{1-z/4} = 3/4 * (1 + (z/4) + (z/4)^2 + ...).$$

$$P(f_2 = n) = 3/4 * (1/4)^n.$$

$$\frac{z+2}{4-z} = 2/4 * (1 + (z/4) + (z/4)^2 + ...) + z/4 * (1 + (z/4) + (z/4)^2 + ...).$$

$$P(f_3 = 1) = 1/2. P(f_3 = n) = 1/2 * (1/4)^n + 1/4 * (1/4)^{n-1}, n > 1.$$

31) I have very little to say regarding this question. It is quite hard. Consider the cases of what suit the dealer has. If he has DIAMONDS or CLUBS, then it is easy to compute the prob. of winning. If he has SPADES or HEARTS, then it gets complicated, as we could draw a heart or spade, respectively. My advice is don't worry about it.

32): The number of such numbers divisible by 10 and 14 is the number of such numbers divisible by $lcm(10, 14) = 70$. Thus the number of such numbers is 3. But we see that:

$$1/10 * 1/14 = 1/140 \neq 3/210.$$

So A and B are not independent. Similarly, the number of numbers divisible by 10 and 15 is the number of numbers divisible by $lcm(10, 15) = 30$. This number is 7. But:

$$1/10 * 1/15 = 1/150 \neq 7/210.$$

So A and C are not independent. But: the number of numbers divisible by 14 and 15 is the number of numbers divisible by $lcm(14, 15) = 210$. Thus there is one number. And:

$$1/14 * 1/15 = 1/210.$$

So B and C are independent.

33): We quickly see that this relation is reflexive, not symmetric, transitive, antisymmetric, not an equivalence relation, and a partial ordering.

34): The largest number of vertices will result when all of them have degree 4. Then:

$$\#edges = \#vertices * 4/2 = 2 * \#vertices = 20.$$

Thus, $\#vertices = 10$.

35): You can see that the following are minimal dominating sets:

$$\{a, h\}, \{e, b, h\}, \{f, g, h\}.$$

36): Pick a person, say person 1. There are 18 other people besides him. So if he sits next to two new people each day, class can last at most $18/2 = 9$ days. The trick is to show that it actually can last 9 days. To see this, let the order on day 1 be:

$$1, 2, 3...19.$$

The order on day 2 be:

$$1, 3, 5, ...19, 2, 4, ...18.$$

The order on day 3 be:

$$1, 4, 7, ...19, 3, 6, ...18, 2, 5, ...17.$$

Up until the order on day 9 is:

$$1, 10, 19, 9, 18, 8, 17, 7, 16, 6, 15, 5, 14, 4, 13, 3, 12, 2, 11.$$

Look at this, and you'll see it works.

37): This is easy once you know what a refinement is. It's just a relation that contains the other relation. So for a), we see that the smallest equivalence relation that is a refinement of R is:

$$aSb, bSa, cSa, aSa, bSb, cSc, dSd, cSb, bSc, aSc.$$

For b), there can be no refinement of R that is a partial ordering, since it would have to satisfy aSb and bSa, thus not be antisymmetric.