Math 55: Discrete Math
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October 20, 1997
Poker Hand Solutions

Note: Remember that aces may be either high or low in poker.

Note: These solutions have been double-checked with Hoyle's Book of Games.

§1: Royal Flush: The 10-J-Q-K-A of any suit.

The only choice we have here is suit. Thus, there are **4** royal flushes.

§2: Straight Flush: A straight of a single suit, not a royal flush.

The straight may be ace low through 10 low. Thus there are 10 choices for the numerical values in the straight. We may choose any suit. Remembering to subtract out royal flushes, we see that there are $4 * 10 - 4 = \mathbf{36}$ straight flushes.

§3: Four of a Kind.

There are going to be two ranks in our hand. We must choose them. There are $\binom{13}{2}$ ways to do this. There are 2 ways to distinguish which of these becomes the quartet. There are $\binom{4}{4}$ ways to choose the suits of the quartet. There are 4 ways to choose the suit of the singleton. Thus there are $\binom{13}{2} * 2 * \binom{4}{4} * 4 = \mathbf{624}$ 4-of-a-kind hands.

§4: Full House: Three of one kind, two of another.

There are two ranks in our hand. There are $\binom{13}{2}$ ways to choose them. We must distinguish one to become the triple. There are two ways to do this. There are $\binom{4}{3}$ ways to choose the suits for the triple, and $\binom{4}{2}$ ways to choose the suits for the double. Thus we have $\binom{13}{2} * 2 * \binom{4}{3} * \binom{4}{2} = \mathbf{3744}$ full houses.

§5: Flush: Five of one suit, not a straight or royal flush.

There are $\binom{13}{5}$ ways to pick 5 different cards. There are 4 ways to pick the suit. We must remember to subtract out royal and straight flushes. Thus there are $\binom{13}{5} * 4 - 4 - 36 = \mathbf{5108}$ flushes.

§6: Straight: Five in a row, not a straight or royal flush.

The straight may be ace low through 10 low, a total of 10 choices. We may choose any suit for each card. Remembering to subtract out straight and royal flushes, we see that there are $10 * 4^5 - 4 - 36 = \mathbf{10200}$ straights.

§7: Three of a kind.

There will be 3 ranks in our hand. There are $\binom{13}{3}$ ways to choose this. There are 3 ways to distinguish which will become the triple. There are $\binom{4}{3}$ ways to choose the suits for the triple, and 4 ways to choose the suit for each singleton. Thus there are a total of $\binom{13}{3} * 3 * \binom{4}{3} * 4^2 = \mathbf{54{,}972}$ threes-of-a-kind.

§8: Two pair.

There are $\binom{13}{3}$ ways to choose the 3 ranks represented in the hand. There are then $\binom{3}{2}$ ways to distinguish which two ranks will become the pairs. There are then $\binom{4}{2}$ ways to choose the suits for each pair, and 4 ways to choose the suits of the singelton, for a total of $\binom{13}{3} * \binom{3}{2} * \binom{4}{2} * \binom{4}{2} * 4 = \mathbf{123{,}552}$ two-pair hands.

§9: One pair.

There are $\binom{13}{4}$ ways to choose the 4 ranks represented in our hand. There are 4 ways to distinguish which will become the pair. There are $\binom{4}{2}$ ways to choose the suits for the pair, and 4 ways to choose the suit of the singletons. Thus there are $\binom{13}{4} * 4 * \binom{4}{2} * 4^3 = \mathbf{1{,}098{,}240}$ one-pair hands.

§10: Nothing.

We could just take $\binom{52}{5}$ minus the sum of all the above numbers, but let us make sure we know what we are doing. There are $\binom{13}{5}$ ways to choose 5 different ranks, and 4 ways to choose the suit of each. Remembering to subtract out royal flushes, straight flushes, flushes, and straights, we see that there are $\binom{13}{5} * 4^5 - 4 - 36 - 5108 - 10200 = \mathbf{1{,}302{,}540}$ hands with nothing.

You can verify that the above numbers indeed sum to $\binom{52}{5}$.