Math 55: Discrete Math
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Old Final Exam Solutions

1) The number of onto functions from a set with 7 elements to a set with 3 elements: You know how to do this one. Just plug right into your formula.

2) Coefficient: It's $\binom{12}{7}2^7(-1)^5$.

3) Pokerhontas: We've done this one several times already. First, since there is only one non-king, if two cards are chosen, one must be a king. Thus, in part a), the desired conditional probability is merely the probability of getting two kings, which is one way out of three total ways, so $\frac{1}{3}$. For part b), there are only two ways to get the king of hearts, and one of them also gets the king of spades. So the desired conditional probability is $\frac{1}{2}$ now.

4) The adjacency matrix: The number of vertices is easy: it's just the number of rows of the matrix, 4. The number of edges is a bit harder: the non-loop edges are each counted for two different vertices, so we have to add-up the off-diagonal terms, and rememeber to divide by 2. This gives us $3+3+2+1+1 = 10$. Now, the loops are each only counted as an edge once, so vertex 4 has 2 edges (loops) attached to it. So the total is $10+2 = 12$.

5) The C++ problem: The easiest way to solve this problem is to compute the probability that it doesn't happen. That is, the probability that we get no C's or less than 2 +'s. Thus, we want to compute the probability of a union of two things, so it's going to involve inclusion-exclusion. The probability of getting no C's is the probability of getting n +'s in n independent choices, so it's $(2/3)^n$. The probability of getting no +'s is the probability of getting n C's, so it's $(1/3)^n$. The probability of getting exactly 1 C is the probability of getting 1 C and $(n-1)$ +'s, so it's $\binom{n}{1}(2/3)^{n-1}(1/3)$. Now, we can't get *both* no C's *and* less than 2 +'s, as then we would have less than 2 cards, but we have at least 3 cards. So the probability of the intersection is 0. Thus the desired probability is:

$$1-((2/3)^n+(1/3)^n+\binom{n}{1}(1/3)^{n-1}(2/3)).$$

6) the book problem: a) if all the books look the same, the only thing that matters is the number of books on each shelf. So this is just a bagel problem, getting n bagels of k kinds. So the answer is:

$$\binom{(n+k-1)}{(k-1)}.$$

b) Now, order does matter. So once we pick the numbers of books on each shelf as in part a), we can have any of the $n!$ orders of the books. So the answer for b) is:

$$\binom{(n+k-1)}{(k-1)}n!.$$

7) the fibonacci problem: There are 3 remainders mod 3, so we have 250 things in 3 boxes. Thus the pigeonhole principle says one box (a remainder mod 3) will have at least $ceiling(250/3) = 84$ numbers in it. For part b), we have to find the exact pattern of the remainders mod 2. We see that mod 2, the fibonacci numbers begin: $0, 1, 1, 0, 1, 1...$ and repeat like that. So if the subscript is divisible by 3, the Fibonacci number is even. So the desired number is the number of numbers between 0 and 249 inclusive, that are divisible by 3. This number is 84.

8) the 55 problem: right off the first midterm. Notice that 55 is not prime, so Fermat's little theorem is irrelevant. We have to use RSA encryption. $p = 5, q = 11, e = 11$. We get that $d = 11$.

9) the induction problem: Follow the hint and you arrive exactly at the desired fact. the induction is also clear.

10) the table of contents problem: (a) The least upper bound of 2 and 8 is 13. (b) The minimal elements are 1 and 7. The maximal elements are 6,14,15. (c) This is just asking for a list of the numbers such that no number comes before a number after it on the chart. the easiest total ordering is just the numbers from 1 to 15 in the usual order.

11) The O question: No, π^x is not $O(10^x)$, as the ratio is $(10/\pi)^x$, which goes to infinity.

12) bit strings of length 12: If the bit string contains at most three 1's, then it contains either 0,1, or 2 1's. So the desired number of bit strings is:

$$\binom{12}{0} + \binom{12}{1} + \binom{12}{2}.$$

13) bit strings of length 10: If the bit string contains at least three 1's and at least three 0's, then it contains either 3,4,5,6, or 7 1's. So the desired number of bit strings is:

$$\binom{10}{3} + \binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7}.$$

14) the sock problem: there are 3 black socks. 2 of them have the property that the sock bundled up with it is also black. So the desired conditional probability is $\frac{2}{3}$.

15) the bagel problem: You know the formula here as well: it's:

$$\binom{(14+5-1)}{(5-1)} = \binom{18}{4}.$$

16) The coin game: The probability that Susan wins the first time through is $(1/2)^3$. The probability that she wins on the second time through is $(1/2)^6$. The probability that she wins on the n^{th} time through is $(1/2)^{3n}$. So the probability that she wins is:

$$\sum_{n=1}^{\infty} (1/8)^n = (1/8)(8/7) = (1/7).$$