Mathematics 55 — Professor K. A. Ribet
Last Midterm Exam — April 8, 1996

Ribet attempts the questions

1 *(9 points).* *A hand of three cards is drawn from an ordinary deck. If it is known that two of the cards are the ace of spades and the ace of clubs, what is the probability that all three cards are aces?*

There are 50 possibilities for the third card, i.e., the card that is not the ace of spades or the ace of clubs. Of these 50, exactly two are aces. Hence the answer is $\frac{2}{50} = \frac{1}{25}$.

2a *(7 points).* *Stan wears either a bathing suit alone, or else a bathing suit, a T-shirt, and a hat. He has 6 bathing suits, 9 T-shirts and 7 hats. In how many different ways can he get dressed?*

There are 6 ways for him to get dressed with only a bathing suit and $6 \times 9 \times 7$ ways for him to get dressed with a bathing suit, T-shirt and hat. So the answer is $6 + 6 \times 9 \times 7 = 384$.

2b *(9 points).* *In how many ways can I divide a class of 15 students into five groups of three students?*

The number of ways to choose "group 1," "group 2," ..., "group 5" is

$$\binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = 168168000.$$

Since the order of the groups doesn't matter, we should divide by 5!. This yields 1401400 as the answer. Alternatively, we can obtain this answer as

$$\binom{14}{2}\binom{11}{2}\binom{8}{2}\binom{5}{2}\binom{2}{2}.$$

The logic for the latter answer was explained in class: pick an arbitrary member of the class, say Hector. He has to be in some group, and there are $\binom{14}{2}$ ways of teaming him up with 2 other students. Once his group is constituted, we have 12 remaining students to be divided into groups of 3. We continue the procedure by picking one of the 12 and completing her group by picking 2 out of the remaining 11 students....

3 *(11 points).* *Find the number of solutions to $x + y + z = 10$ in non-negative integers which satisfy $x \le 5$ and $y \le 3$.*

There are 6 possibilities for x and 4 possibilities for y. You can take (x, y) arbitrarily, and then put $z = 10 - x - y$. So the answer is 24.

Now, here's the plodding way to do it. Let S be the set of all solutions to $x + y + z = 10$ in non-negative integers. It has $\binom{12}{2} = 66$ elements. (You recognize the bagel problem here, but Noah's is closed for Passover, so I didn't mention bagels.) Let A be the set of solutions with $x \ge 6$; it has $\binom{6}{2} = 15$ elements because it's the same as the set of all solutions to $a + y + z = 6$. Let B be the set of solutions to $x + y + z = 10$ with $y \ge 4$. Similarly, it has $\binom{8}{2} = 28$ elements. The set $A \cap B$ has only one element: the solution $(6, 4, 0)$. Hence the number of elements of $A \cup B$ is $15 + 28 - 1 = 42$ (inclusion-exclusion). The question asks for the set of solutions *not* in $A \cup B$. This has $66 - 42 = 24$ elements.

4 *(9 points).* *A hand of three cards is drawn from an ordinary deck. If it is known that at least two of the cards are aces, what is the probability that all three cards are aces?*

The number of hands with 3 aces is $\binom{4}{3} = 4$. The number of hands with 2 aces and a non-ace is $\binom{4}{2} \times \binom{48}{1} = 288$. The probability that a hand with at least two aces is a 3-ace hand is thus $4/(4 + 288) = 1/(73)$.