Mathematics 55 Professor K. A. Ribet
First Midterm Exam February 26, 1996

Ribet's answers to the questions

1a *(5 points). True or false: If a function f maps a set to itself, then f is one-to-one if and only if f is onto.*

This is patently false. For example, the function $x \mapsto 2x$ on the set of positive integers is 1-1 but not onto.

1b *(5 points). True or false: If p and q are propositions, then $\neg p \to q$ and $p \wedge q$ are equivalent.*

This seems at first to be true because $\neg p \to q$ is equivalent to $p \vee q$. However, it is false! Namely, if exactly one of p and q is true, then $\neg p \to q$ is `T` but $p \wedge q$ is `F`.

2a *(5 points). In the sleepy town of Cognac, 33 cars have license plates containing* `TI`*, while 109 have license plates containing the string* `NY`*. Suppose that 115 cars have license plates containing either* `TI` *or* `NY`*. How many cars have plates with* `TI` *or* `NY`*, but not both?*

This is a simple application of inclusion-exclusion. Let A be the set of cars with `TI` and let B be the set of cars with `NY`. What is given is that $A \cup B$ has order 115. Hence $A \cap B$ has order $33 + 109 - 115 = 27$ by the inclusion/exclusion formula for two sets. The question asks for the number of objects in $A \cup B$ but not in $A \cap B$. The answer is then $115 - 27 = 88$.

2b *(8 points). Use the Euclidean algorithm to solve the congruence $5x \equiv 1$ (mod 13).*

Well, $13 = 2 \times 5 + 3$, $5 = 1 \times 3 + 2$, $3 = 1 \times 2 + 1$. Hence:

$$1 = 3 - 2 = 3 - (5 - 3) = 2 \times 3 - 5 = 2 \times (13 - 2 \times 5) - 5 = 2 \times 13 - 5 \times 5.$$

Thus $(-5) \cdot 5 \equiv 1 \pmod{13}$. Hence "$-5$" is the answer. Another possible answer is 8, since -5 and 8 are the same mod 13. The most complete answer is to say "-5 plus any multiple of 13."

3 *(10 points). The celebrated Ribonacci numbers R_n are defined as follows:*

$$R_0 = 0; \quad R_1 = 1; \quad R_n = 3R_{n-1} - 2R_{n-2} \text{ for } n \geq 2.$$

Prove that $R_n = 2^n - 1$ for all $n \geq 0$.

This is clearly going to be a proof by induction. The only wrinkle is that you can't run the simplest possible induction, in which the nth assertion implies the one after it. We take $P(n)$ to be the assertion "$R_n = 2^n - 1$." It is immediate that $P(0)$ and $P(1)$ are true. Moreover, $P(n)$ is implied by the conjunction of $P(n-1)$ and $P(n-2)$. Indeed, if $R_{n-1} = 2^{n-1} - 1$ and $R_{n-2} = 2^{n-2} - 1$, then

$$R_n = 3(2^{n-1} - 1) - 2(2^{n-2} - 1) = 6 \cdot 2^{n-2} - 2 \cdot 2^{n-2} - 1.$$

Since $6 - 2 = 4 = 2^2$, we get the desired formula $R_n = 2^{n-2+2} - 1$. Thus, the induction principle shows that $P(n)$ is true for all n.

4a *(5 points). True or false: If $f(x) = \log_\pi x$ and $g(x) = \log_{10} x$, then $f(x) = O\big(g(x)\big)$.*

Sounds like it's true. The ratio $f(x)/g(x)$ is a constant, since $\log_\pi x = (\ln x)/(\ln \pi)$ and $\log_{10} x = (\ln x)/(\ln 10)$. To satisfy the definition of "O," we can take C to be this constant and K to be anything.

4b *(7 points). Find a non-negative integer x satisfying the two congruences*

$$\begin{cases} x \equiv 15 & \text{(mod 18)} \\ x \equiv 47 & \text{(mod 50).} \end{cases}$$

[Hint: first find a solution with x negative.]

The negative value of x is $x = -3$; this clearly works! To find a positive value, it suffices to add $18 \times 50 = 900$ to x, getting 897. In fact, if x is any solution, then so is $x + t$ when t is divisible both by 50 and by 18. So we could take $t = 450$, getting the smaller answer 447 to this problem.