Math 55: Discrete Math
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Midterm #2 Solutions

#1a) How many poker hands contain no ace, exactly one king, and at least one heart?

This problem was one of the most difficult to grade because of the multitude of creative ways students find in approaching the problem. The simplest way to solve this is to realize that to construct a hand with no ace and one king, first pick the king (4 ways), and then the other 4 cards ($C(44,4)$ ways). Out of these, the number of hands containing no hearts can be found the same way: first choose a king (3 ways), and then the other 4 cards ($C(33,4)$ ways). So the answer is

$$4 \cdot \binom{44}{4} - 3 \cdot \binom{33}{4}.$$

Other popular correct answers include

$$\binom{44}{4} + 3(\binom{44}{4} - \binom{33}{4})$$

and

$$\binom{44}{4} + 3(\binom{11}{1}\binom{33}{3} + \binom{11}{2}\binom{33}{2} + \binom{11}{3}\binom{33}{1} + \binom{11}{4}\binom{33}{0}).$$

#1b) How many different strings of length ten can one make out of the letters in INDISCRETE?

The number of permutations of these ten letters in 10!, but this includes switching the Is and switching the Es. Therefore, this figure must be divided by $(2!)^2$. The answer is then

$$\frac{10!}{4}.$$

#2) How many solutions to $x + y + z + w = 1097$ in non-negative integers $x, y, z, and w$ satisfy at least one of the inequalities $x \geq 100$, $y \geq 100$, $z \geq 100$?

If we ignore these inequalities, the stars-and-bars method of bagel-picking tells us that the number of solutions is $C(1100, 3)$. How many of these solutions do *not* satisfy any of the inequalities? If $x < 100$, $y < 100$ and $z < 100$, there are 100 choices for x, 100 for y and 100 for z.

Furthermore, given any x, y, z all less than 100, the equation is solved by setting $w = 1097 - (x + y + w) > 0$. Therefore, since there are $100 \times 100 \times 100$ solutions which do not satisfy any of these inequalities, the answer is

$$\binom{1100}{3} - 1,000,000.$$

Another popular correct answer is

$$3\binom{1000}{3} - 3\binom{900}{3} + \binom{800}{3}.$$

#3) In how many ways can a class of 15 be divided into 5 groups of 3 students in such a way that the two students named Ken are in the same group?

Since the Kens are on a team of three, they need another teammate. Pick one from the 13 students left (13 ways). For each choice, there are now 12 students remaining, who must be divided into 4 teams of 3. By the method demonstrated in class, we see that there are $C(11,2) \cdot C(8,2) \cdot C(5,2)$ ways to do this. Therefore the answer is

$$13 \cdot \binom{11}{2} \cdot \binom{8}{2} \cdot \binom{5}{2}.$$

#4a) Persi's crooked penny comes up "heads" 2/3 of the times when it is tossed. What is the probability that exactly four heads come up when it is tossed six times?

First, we find the number of heads-tails strings of length six containing exactly four heads. This is just $C(6,4)$. Then given such a string, the probability of flipping the coin to match this string is $(2/3)^4(1/3)^2$. Therefore, the answer is

$$\binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2.$$

#4b) A fair nickel and fair penny are tossed simultaneously until both come up "tails." What is the expected number of tosses?

Let T be the random variable representing the number of tosses, let p be the probability of getting both tails on a given toss, and let H be the random variable representing the number of times we get both tails.

Then $E(T) \cdot p = E(H)$. We know also that $E(H)$ is 1, since we stop once we get both tails. Also, it is clear that on a given toss, the probability of each coin showing tails is $1/2$, so $p = 1/4$. It follows then that $E(T) = 4$, so the answer is

$$4.$$

#5a) What is the coefficient of $x^{30}y^{29}$ in the expensions of $(x-2y)^{59}$?

By the binomial formula, we know that

$$(x-2y)^{59} = \sum_{i=0}^{59} \binom{59}{i} x^i (-2y)^{(59-y)}.$$

The term with $x^{30}y^{29}$ is then found by setting $i = 30$. This term is then

$$\binom{59}{30} x^{30}(-2y)^{29} = \left(\binom{59}{30}(-2)^{29}\right) x^{30}y^{29}$$

so the answer is

$$\binom{59}{30}(-2)^{29}.$$

#5b) How many integers are needed to guarantee that two of them leave the same remainders on division both by 15 and by 21?

By the Chinese Remainder Theorem, we have $x \equiv y \pmod{15}$ if and only if $x \equiv y \pmod 3$ and $x \equiv y \pmod 5$. Furthermore we also know that $x \equiv y$ (mod 21) if and only if $x \equiv y \pmod 3$ and $x \equiv y \pmod 7$. So two numbers leave the same remainders upon division by 15 and 21 if they are congruent modulo 3, 5, and 7.

Then by using Chinese Remainder Theorem again, we know that this only happens when the two numbers are congruent to each other modulo $3 \cdot 5 \cdot 7 = 105$. So the question is asking how many integers are needed to guarantee that two of them leave the same remainder on division by 105.

Since there are 105 possible remainders, we know if we have 106 numbers, then by the Pigeonhole Principle we know that two of them are the same. So the answer is

$$106.$$