Math 55: Discrete Math

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Final Exam Review Sheet

1. An explorer is captured by a group of cannibals. There are two types of cannibals–those that always tell the truth and those that always lie. The cannibals will barbecue the explorer unless he can determine whether a particular cannibal always lies or always tells the truth. He is allowed to ask the cannibal exactly one question.

a) Explain why the question "Are you a liar?" does not work.

b) Find a question that the explorer can use to determine whether the cannibal always lies or always tells the truth.

2. Let $F(x, y)$ be the statement x can fool y, where the universe of discourse is the set of all people in the world. Use quantifiers to express each of the following statements.

a) Everybody can fool Larry.

b) Carmen can fool everybody.

c) Everybody can fool somebody.

d) There is no one who can fool everybody.

e) Everyone can be fooled by somebody.

f) No one can fool both Barry and Jerry.

g) Emmit can fool exactly two people.

h) There is exactly one person whom everyone can fool.

i) No one can fool himself or herself.

j) There is someone who can fool exactly one person besides himself or herself.

3. What is the cardinality of the set $\{a, b, \{b\}, \{b, c\}, \{\{c\}\}\}$?

4. What can you say about sets A and B if

a) $A \cup B = A$?

b) $A \setminus B = A$?

c) $(A \setminus B) \cup (B \setminus A) = \emptyset$?

d) $(A \cap B) \cup (A \setminus B) \cup (B \setminus A) = A \cup B$?

5. Let f be a function from the set A to the set B. Let T and S be subsets of A. Show that $f(S \cup T) = f(S) \cup f(T)$ and $f(S \cap T) \subset f(S) \cap f(T)$.

6. Compute the value of the double sum

$$\sum_{i=0}^{3}\sum_{j=0}^{2}(3i+2j).$$

7. Show that if $f(x)$ is $O(x)$ then $f(x)$ is $O(x^2)$.

8. Find a function that is $O(x \log x)$ but not $O(x)$.

9. **a)** Find gcd(1044, 598) using the Euclidean algorithm.

b) Find integers m and n such that

$$1044m + 598n = \gcd(1044, 598).$$

10. Find *all* the solutions to the following system of congruences:
$x \equiv -1 \pmod 6$
$x \equiv 15 \pmod{25}$
$x \equiv 10 \pmod{11}$

11. Consider the following system of congruences:
$x \equiv -3 \pmod{18}$
$x \equiv 13 \pmod{15}$

a) Does this have a solution? If so, find it. If not, explain why not.

b) How about this system?
$x \equiv 1 \pmod{18}$
$x \equiv 1 \pmod{15}$

12. Use RSA-encryption to encode the number 414, using $p = 23$, $q = 29$, and $e = 3$. Decode the encrypted number.

13. What is the smallest prime number that divides $2310^{144} - 1$?

14. If S is a set of ten natural numbers between 1 and 99 inclusive, show that there are two disjoint non-empty subsets of S with equal sums of their elements.

15. Show that $n^5/5 + n^4/2 + n^3/2 - n/30$ is an integer for any natural number n.

16. Show that for any set of n integers, there is a non-empty subset of them whose sum is divisible by n.

17. Use induction to prove that every positive integer can be written as the sum of distinct Fibonacci numbers.

18. Show that at a party with at least two people, there are two people who know the same number of other people there. (Assume that if a knows b, then b knows a.)

19. Show that any subset of 55 distinct positive numbers less than 101 must contain two numbers which differ by exactly 9.

20. Use induction to show that 21 divides $4^{n+1} + 5^{2n-1}$ for any positive integer n.

21. How many strings of 6 distinct lowercase letters from the English alphabet contain

a) the letters a and b?
b) the letters a and b in consecutive positions?
c) the letters a and b, with a somewhere to the left of b?

22. How many different strings of length 13 can be formed using the letters in MATHEMATICIAN, if the two I's must be consecutive?

23. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 27$$

such that $x_1 \geq 2$, $x_2 \leq 4$, $x_3 \geq 6$, and $x_4 \leq 8$?

24. How many positive integers not exceeding 1000 are divisible by at least one of 6, 10, or 14?

25. What is the probability that an even number of people receive the correct hat from a hatcheck person who gives 10 people their hats back at random?

26. Let $A = \{1, 2, 3, \ldots, 10\}$ and let $B = \{1, 2, 3\}$. How many functions $f : A \to B$ are there such that each element of B is the image under f of at least **two** elements of A?

27. A standard cubic die and a funky octahedral die (with 8 sides) are rolled simultaneously. The faces on the octahedron are marked with the numbers between 0 and 7. Let f be the sum of the values showing on the two dice. Find $E(f)$ and $V(f)$.

28. Woody has a bag of blocks. The blocks have three shapes: squares, circles, and triangles. They also come in three colors: red, blue, and green. Woody declares that if a block is drawn out of his bag at random, the probability of it being red is greater than the probability of it being blue. Furthermore, he continues, the probability that it is a square is greater than the probability of it being a circle. Yet he also maintains that the block is more likely to be a blue circle than a red square.

a) Prove that for a random block, the events of being red and being a square are *not* independent.

b) Woody then tells us that there are exactly five blocks in the bag. What are they?

29. A number between 1 and 10000 is chosen at random. What is the probability that this number is a palindrome? [Remember: a palindrome is a number written the same backwards, such as 8, 737, or 1221.]

30. Let f_1 and f_2 be independent random variables whose images are contained in the non-negative integers, and let

$$g_1(z) = \frac{z+2}{3} \text{ and } g_2(z) = \frac{3}{4-z}$$

be their respective generating functions.

a) Find the generating function for $f_3 = f_1 + f_2$.

b) Calculate $E(f_1)$, $E(f_2)$, and $E(f_3)$.

c) Calculate $V(f_1)$, $V(f_2)$, and $V(f_3)$.

d) For arbitrary n find $P(f_1 = n)$, $P(f_2 = n)$, and $P(f_3 = n)$.

31. At a local casino, a game is played called "Suit's Me!" A deck of 52 cards is shuffled and one card is dealt to the dealer and one to the player.

a) If the suits match, this "suits you," and the player wins.

b) If the suits don't match, but the colors do, the dealer wins.

c) If the suits are of different colors, then whoever's suit has more letters wins.

d) If the suits have the same number of letters, the player is given another card.

Find the probability of winning this game. (Like all casino games, your answer should be slightly less than one half.)

32. A number n is chosen at random between 1 and 210. Which of the following events are independent?

A) $10|n$ B) $14|n$ C) $15|n$

33. Consider the following relation on the set of graphs. Let G_1 and G_2 be two graphs. Then $G_1 R G_2$ if G_1 is a subgraph of G_2. Is R:

a) reflexive?

b) symmetric?

c) transitive?

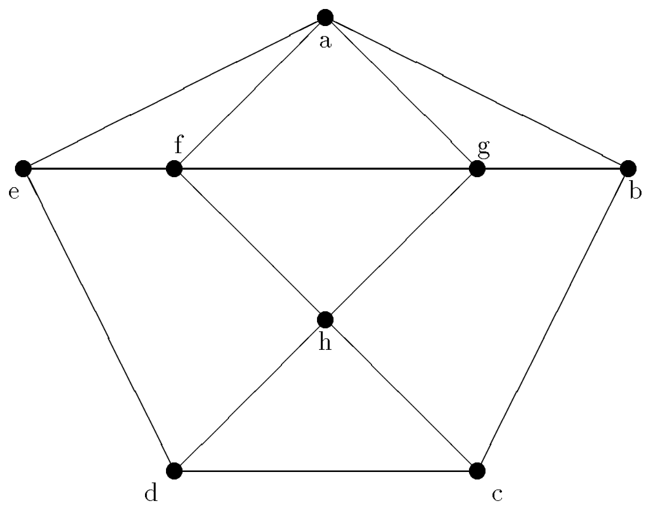
d) antisymmetric?

e) an equivalence relation?

f) a partial ordering?

34. What is the largest possible number of vertices in a graph with 20 edges if each vertex has degree at least 4?

35. A set of vertices is said to *dominate* a graph if every vertex in the graph is either in the set or adjacent to a vertex in the set. A *minimal dominating set* of vertices is a set which dominates the graph, such that no proper subset dominates the graph. Find minimal dominating sets of different sizes for the following graph:



36. Math 55 has 19 students, who sit at a circular table. The instructor, Ken Rosen, insists that each day students sit next to two new people. How many days can class last?

37. Let R be a relation on the set $\{a, b, c, d\}$ such that aRb, bRa, and cRa.

a) Find the smallest refinement of R that is an equivalence relation on $\{a, b, c, d\}$.

b) Find the smallest refinement of R that is a partial ordering on $\{a, b, c, d\}$.

GOOD LUCK ON THE FINAL!!!