Mathematics 55 — Professor K. A. Ribet
Final Exam — December 17, 1997

Ribet attempts the questions

These solutions were prepared by me, Ken Ribet, on December 17. I apologize in advance for any errors that I've written down. I hope that, despite the inevitable errors, you will find this solution set helpful. Merry XMAS, all!

1a *(5 points). How many subsets of the set $\{1, 2, \ldots, 19\}$ contain at least one odd integer?*

There are 2^{19} subsets of the indicated set. The ones that *don't* contain any odd integers are subsets of the set $\{2, 4, 6, 8, \ldots, 18\}$, which has 9 elements. So there are 2^9 subsets of the big set that contain no odd integers. The answer is thus $2^{19} - 2^9$.

1b *(5 points). Find the last (rightmost) decimal digit of 3^{1024}.*

The number in question is $(81)^{256}$. Since 81 is 1 mod 10, so is the given number. In other words, the answer is 1.

2a *(5 points). If $P(A) = 1/3$, $P(B) = 1/2$ and $P(A \cup B) = 2/3$, find $P\left((A \cap B)|A\right)$.*

The probability of the event $A \cup B$ is $P(A) + P(B) - P(A \cap B)$. Hence $P(A \cap B) = 1/6$. The answer to the question is $P(A \cap B)/P(A)$, which is $3/6$, or $1/2$.

2b *(5 points). Mumford has a trick quarter with two heads, a trick quarter with two tails, and a standard quarter with one head and one tail. He chooses one of the three coins at random, tosses it in the air, and slaps it on the table. A head is showing. Find the probability that the selected coin has two heads.*

I view Mumford's procedure as that of choosing at random one of six possible coin-sides and displaying it to the world. (There are 3 coins, each with 2 sides.) We are given that the displayed side is a head, so our sample space is reduced to the three-element set consisting of the two sides of the quarter with two heads and the head in the non-trick coin. In this universe, two of the three heads have heads on the back. So my answer here is $2/3$.

3 *(9 points). For each positive integer n, let $f(n)$ be the number of subsets of $\{1, 2, \ldots, n\}$ which contain no two consecutive integers.*

(a) Calculate $f(1)$, $f(2)$ and $f(3)$.

(b) Find a recursive formula for $f(n)$ and use it to calculate $f(8)$. [Justify your recursive formula.]

Clearly $f(1) = 2$, since no subsets of $\{1\}$ contain a pair of consecutive integers. Also, $f(2) = 4 - 1 = 3$, since only the full set $\{1, 2\}$ contains a pair of consecutive integers. Similarly, $f(3) = 5$, since we have to avoid $\{1, 2, 3\}$, $\{1, 2\}$ and $\{2, 3\}$. The numbers 2, 3 and 5 are meant to evoke the Fibonacci numbers. If you realize that $f(n) = f(n-1) + f(n-2)$, i.e., that the pattern will persist, then you are forced to conclude that $f(8) = 55$. How to justify the recursive formula? The point is that a subset of $S_n := \{1, 2, \ldots, n\}$ either contains n or it doesn't. If it doesn't, it's just a subset of S_{n-1} and the property of not containing consecutive integers gets transferred to the $(n-1)$-level. If it does contain n, then it doesn't contain $n - 1$, and we're dealing with a set which is made by adding n to a subset of S_{n-2} which contains no pair of consecutive integers.

4 *(8 points).* *According to* `math.berkeley.edu`*, the number of Math 55 students who visited the class Web page at least once in November is divisible by 1, 2, 3, 4, 5 and 6, but leaves remainder 1 on division by 7. What is this number?*

Rest assured, `math.berkeley.edu` doesn't know who is downloading its Web pages. The problem doesn't correspond to reality. Let's take it on its own terms, though and let N be the number in question. It's a multiple of the least common multiple of the numbers 1–6, which is 60. The possibilities for N are thus 0, 60, 120, 180, 240, 300: you can rule out higher numbers by looking around the room and seeing that there aren't more than 300 students in the class. The number 120 is the only one here which is 1 more than a multiple of 7: $120 = 17 \cdot 7 + 1$, and any other number which is divisible by 60 and 1 mod 7 is congruent to 120 mod 420. So the answer is 120. If you don't want to fool around with trial divisions, you are in the business of solving the congruence $60x \equiv 1 \bmod 7$, and you can do this in a very routine algorithmic way....

5 *(9 points).* *Show that*

$$\sum_{i=1}^{2n}(-1)^{i+1}\frac{1}{i} = \sum_{i=n+1}^{2n}\frac{1}{i}$$

for all positive integers n.

This is an induction problem, as was supposed to be fairly obvious to you from the formulation. For $n = 1$, it says that $1 - 1/2 = 1/2$, which is certainly ok. If you compare the statement for n with the statement for $n + 1$, then the left side acquires two extra terms $1/(2n+1) - 1/(2n+2)$, while the right side acquires an extra $1/(2n+1) + 1/(2n+2)$ but *loses* $1/(n+1)$. You have to see then that

$$\frac{1}{2n+1} - \frac{1}{2n+2} \stackrel{?}{=} \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1},$$

which is clearly OK.

You might ask why a formula like this might be interesting. As $n \to \infty$, the left side obviously approaches the sum of the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} + \cdots$. The right side looks to me like a Riemann sum for the integral $\int_1^2 \frac{1}{x}\,dx = \ln 2$. Once you know that the right-hand side approaches $\ln 2$, you have a proof that the sum of the alternating harmonic series is $\ln 2$. In Math 1, we prove that the alternating harmonic series converges, and we say that the sum is $\ln 2$ without giving a proof of this fact. This problem might be viewed as an intelligent way to justify the fact that the sum is indeed $\ln 2$.

6 *(9 points).* *A box contains three red socks, three blue socks, and four white socks. (Socks of the same color are indistinguishable.) Eight socks are pulled out of the box, one at a time. In how many ways can this be done?*

As we shouted out in the exam room, order matters here. If order didn't count, we'd be looking at the number of solutions to $R + B + W = 8$ with $R \le 3$, $B \le 3$ and $W \le 4$. There are six of them. One way to see this is to write $x = 3 - R$, $y = 3 - B$, $z = 4 - W$. Then $x + y + z = 2$, and we're in bagel land. The solutions to the latter problem (with x, y and z) number six: $(1,0,1)$, $(0,1,1)$, $(1,1,0)$, $(2,0,0)$, $(0,2,0)$, $(0,0,2)$. In terms of R,

B, and W, the solutions are $(3,3,2)$, $(2,3,3)$, $(3,2,3)$, $(3,1,4)$, $(2,2,4)$, and $(1,3,4)$. Now, think about order. If there are 3 red, 3 blue and 2 white socks, then the number of ways of pulling these out of a drawer, in some order, is $8!/(3!3!2!)$. The number would be the same if you had 2 red, 3 blue and 3 white socks. Etc. So the final answer is

$$3\frac{8!}{3!3!2!} + 2\frac{8!}{3!1!4!} + \frac{8!}{2!2!4!} = 2660.$$

I'm not sure of the 2660, which I calculated fast.

7 *(10 points).* *Let G be the simple graph whose vertices are the bit strings of length 6, two bit strings being connected by an edge if and only if they differ in exactly one place.*

(a) Does G have an Euler circuit?

(b) Is the graph G planar?

It seems to me that there are 6 edges coming out of every vertex, since you can change one bit in exactly 6 ways. The degree of each vertex is 6, which is even, so there is an Euler circuit. The total number of edges is one-half $6 \cdot 2^6$, or 3 times the number of vertices. For a planar graph, you have to have $e \leq 3v - 6$, but here we have $e = 3v$, so the graph isn't planar.

8 *(8 points).* *Suppose that A is a finite set with at least two elements and that R is an equivalence relation on A (for instance, the relation of acquaintance among guests at a party). Show that there are distinct elements a and a' in A whose equivalence classes $[a]_R$ and $[a']_R$ have the same number of elements.*

As the wording of the problem suggests, I was thinking of the party problem on the review sheet. The problem is true, more generally, for a symmetric, reflexive relation which is not necessarily transitive. As was pointed out during the exam, the acquaintance relation at a party is *not* an equivalence relation; so we threw out the parenthetical hint. To do the problem, which is very easy as stated, ask yourself whether there are any two distinct as which are related. If so, then there are a and a' with $a \sim a'$. Then $[a]_r$ and $[a']_R$ are equal, so they have the same number of elements. If not, $[a]_R$ has one element for all a. Then you can take any pair a, a' of distinct elements in A; their equivalence classes have the same number of elements: 1.

9 *(9 points).* *I roll a single die repeatedly, until three different faces have come up at least once. What is the expected number of times that I need to roll the die?*

This problem was given in the Lenstra notes homework, and a (somewhat heuristic) solution was given in the "homework solutions." (It's the FLAKEY problem in the Lenstra notes.) Suppose that I want only two different faces to come up. Then I roll the die a first time, getting a 3 (say). After the 3 has come up, I'm rolling to get a number different from 3. The probability of getting such a number on a single roll is $5/6$, so the expected number of rolls to get such a number is $6/5$. Thus the total number of rolls is expected to be $1 + 6/5$. "Similarly," once we have two different numbers under our belt, we're rolling to get a number different from the two that we have. The probability of getting such a number on a given roll is $4/6$, so the expected number of extra rolls to get a third distinct number is $6/4$. Hence the answer to the problem is $1 + 6/5 + 6/4$. (Note: an infinite series should not count as a complete answer to this question.)