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A large number of people—100, say, or n to be general—enter a night club carrying umbrellas. In how many ways can the umbrellas be returned to them at the end of the evening?

Answer: $n!$, the number of bijections from $\{1,\ldots,n\}$ to itself. (Define $f(i)=j$ if i's umbrella ends up in j's hands.)

In how many ways can the umbrellas be returned in such a way that no one gets their own umbrella back? Complete scramblings like this are called *derangements*.

Let $D(n)$ be the number of derangements of n umbrellas. What is the *probability* that no one gets his or her umbrella back?

The answer is $\dfrac{D(n)}{n!}$.

As n gets big, what happens to this ratio? Does it become more or less likely that there will be a complete scrambling?

Let $\mathcal{U}$ be the universe of all 1-1 onto functions from $\{1,\ldots,n\}$ to itself. Let A_1 be the set of those bijections $f\in\mathcal{U}$ such that $f(1)=1$. Define A_2, A_3, and so on similarly.

Then $A_1\cup A_2\cup\cdots\cup A_n$ is the set of those bijections which send some i to itself. The complete scramblings are the bijections in the *complement* of this union.

To find $\#(A_1\cup\cdots\cup A_n)$, we use inclusion-exclusion. We have to think about the general formula, which has n terms.

Let's think about the tth term. Aside from its sign, $(-1)^{t+1}$, it is the sum of the sizes of the various intersections of t different sets taken from the A_i. This sum has $\binom{n}{t}$ terms because there are that many possible intersections.

A given term in the sum represents the number of bijections $f\in\mathcal{U}$ which fix some specific set of t different elements in $\{1,\ldots,n\}$. This number is the number of 1-1 onto maps from a set of $n-t$ elements to itself, or $(n-t)!$.

Putting 2 and 2 together, we learn that the tth term in the inclusion-exclusion formula is $(n-t)!\binom{n}{t}=n!/t!$.

Thus the number of elements in the union is

$$\frac{n!}{1!}-\frac{n!}{2!}+\frac{n!}{3!}-\cdots+(-1)^{n+1}\frac{n!}{n!}.$$

Hence

$$\begin{aligned}D(n)&=\frac{n!}{2!}-\frac{n!}{3!}+\cdots+(-1)^n\frac{n!}{n!}\\&=\frac{n!}{0!}-\frac{n!}{1!}+\frac{n!}{2!}-\frac{n!}{3!}+\cdots+(-1)^n\frac{n!}{n!}.\end{aligned}$$

Thus the "probability" of getting all the umbrellas wrong is

$$\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}.$$

To see what happens to this number for n pretty big, we recall from Math 1B that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Thus

$$\frac{1}{e} = e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots.$$

Thus our probability is the nth Taylor approximation to $e^{-1} \approx 0.37$. As you remember from Math 1B, the series converges very fast, so the approximation is excellent as soon as n is bigger than 4 or 5.

In summary: take a group of people that is not ridiculously small. Scramble their umbrellas. The probability of a complete derangement is about 37%; it's pretty much independent of the size of the group! I would deem this to be a surprising result.

What's with the probability business? In general, we could consider a finite sample space S of possible outcomes. An *event* is a subset E of S. To E we attach the probability

$$p(E) = \frac{\#E}{\#S}.$$

In the example, E was the set of derangements, and S was the "universe" $\mathcal{U}$.

Another example: 36 students are divided into teams with 9 members each (baseball?). What is the probability that Sam and Sara end up on the same team?

It might seem natural to let S be the set of ways of dividing 36 students into four 9-person teams. Then

$$\#S = \frac{\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9}}{4!},$$

which is numerically 893864677761055000. The set E should be the subset of S consisting of those team-selections that make team-mates out of Sam and Sara.

Can you see a good way to calculate $\#E$?

Use the asymmetrical method of calculating $\#S$ that we introduced last time. It gives

$$\#S = \binom{35}{8}\binom{26}{8}\binom{17}{8}\binom{8}{8}$$

(same numerical value). To make the first term, we can choose Sara arbitrarily and decree that her team will get selected first. There are 35 non-Saras to choose from and 8 slots for the chosen non-Saras.

The number of ways of choosing Sara's team in such a way that Sam gets put on the team is $\binom{34}{7}$. Hence the probability to be calculated is $\binom{34}{7}/\binom{35}{8} = 8/35$.

Explanation: the set S has $\binom{35}{8}N$ elements, where N is an integer which doesn't interest us much: the product of the final three factors in the formula for $\#S$ that doesn't involve dividing by 4!. The number N is the number of ways of making three 9-person teams out of a set of 27 players.

On the other hand, $\#E$ is $\binom{34}{7}N$, with the same N. After Sara's team is made up, there are still 27 players left to assign and still N ways to do this.