Announcement: I won't be holding office hours tomorrow. (I'm going to Caltech for a day.)

In this lecture I will be talking about *equivalence relations* on a set A. We say that $R \subseteq A \times A$ is an equivalence relation if it is symmetric, reflexive and transitive. When R is an equivalence relation and (a, b) lies in R, we write $a \sim_R b$ (or just $a \sim b$) and say that a and b are equivalent (relative to R). To make an equivalence relation on A is to decide what things will be equivalent.

Examples abound. The relation of being equal is an equivalence relation on any set. The subset R in question is the diagonal:

$$\{ (a, a) \mid a \in A \}.$$

At the other extreme, we could take $R = A \times A$. Here, all elements are equivalent to each other.

A very general example runs as follows. Start with a function $f\colon A \to T$, where T is some set. You declare that a and b in A will be equivalent if $f(a) = f(b)$. The set R is the set of pairs (a, b) with $f(a) = f(b)$. If you take $f\colon A \to A$ to be the identity function, you get the first (diagonal) example. If you take $f\colon A \to \{1\}$ to be the constant function, you get the second (stupid) example.

A special case occurs when you take $A = \mathbf{Z}$ to be the set of integers and pick an integer $m \geq 1$. Let $T = \{ 0, 1, \ldots, m-1 \}$ and let $f\colon \mathbf{Z} \to T$ be the function that takes any integer a to the remainder on dividing a by m. Then $f(a) = f(b)$ if and only if $a \equiv b \bmod m$. Congruence mod m is an equivalence relation — it arises from the function $a \mapsto (a \bmod m)$.

A fundamental fact is that the "very general" example is in some sense the only example. Given an equivalence relation on A, you can fabricate a set T and a function $f\colon A \to T$ so that R arises from f.

Here's how. Assume that R is given. For each $a \in A$, let $[a]$ be the set of things in A which are equivalent to a (under R). Thus $[a]$ is a subset of A. It's called the equivalence class of a. We can let T be the power set of A (i.e., set $T = \wp(A)$) and consider the function

$$f\colon A \to \wp(A)$$

which maps a to $[a]$. The claim is that $a \sim b$ if and only if $[a] = [b]$.

Let's verify this. Assume first that $a \sim b$. If x lies in $[a]$, then x is equivalent to a, which in turn is equivalent to b. By transitivity, x is equivalent to b, so x lies in $[b]$. Thus $[a] \subseteq [b]$, and we get similarly that $[b]$ is a subset of $[a]$. Hence $[a] = [b]$.

Conversely, assume that $[a] = [b]$. Clearly, a lies in $[a]$ by reflexivity, so a lies in $[b]$, which means that $a \sim b$.

Thus we do indeed have that $a \sim b$ if and only if $f(a) = f(b)$.

Here's another crucial fact. Suppose that we have an equivalence relation on A and that we define the equivalence classes $[a]$ as we have been doing. Then any two equivalence classes $[a]$ and $[b]$ are either equal (as subsets of A) or disjoint (i.e., have no elements in common).

Indeed, suppose that $[a]$ and $[b]$ are not disjoint, so that they have an element x in common. If $t \in [a]$, then we have $t \sim a$. We have then

$$t \sim a, \quad a \sim x, \quad x \sim b,$$

so we get $t \sim b$ by transitivity. This means that t is in $[b]$, and we may conclude $[a] \subseteq [b]$. Similarly, we have $[b] \subseteq [a]$, so the conclusion is the required equality $[a] = [b]$.

We have just seen that the different sets $[a]$ which occur are disjoint from each other. Also, they cover A, since every $a \in A$ belongs to one of them (namely, to $[a]$). One says that the equivalence classes (relative to R) form a *partition* of A.

Example: let R be the relation "congruence mod 3" on $\mathbf{Z}$. There are three equivalence classes: (1) the set of numbers which are 1 mod 3, (2) the set of numbers which are 2 mod 3; (3) the set of numbers which are 0 mod 3 (or 3 mod 3). The set $\mathbf{Z}$ is the disjoint union of these three subsets of $\mathbf{Z}$.

The final wrinkle is that you can make up an equivalence relation on A any time that A has been partitioned into a disjoint union of subsets. Suppose that $A = A_1 \cup \ldots \cup A_t$, where $A_i \cap A_j =$ if $i \neq j$. (The partition could have been infinite, but we're looking only at the finite case for simplicity.) We assume that the sets A_i are all non-empty.

Define $f\colon A \to \{1, \ldots, t\}$ so that $f(a) = i$ if i is the unique index so that $a \in A_i$. The corresponding equivalence relation has $a \sim b$ if and only if a and b lie in the same A_i. Hence the equivalence class of any element of A_i is the set A_i, and the different equivalence classes for the relation are the sets A_i.