Announcement: I won't be holding office hours this Friday. (I'm going to Caltech for a day.)

Now we begin the third and final portion of the course: we concentrate on graphs and relations.

A relation from A to B is nothing other than a subset of $A \times B$, the set of pairs (a, b) with $a \in A$ and $b \in B$. It's easy to imagine lots of relations. For example, A and B can be the set of positive integers, and the relation might be the set R of pairs (a, b) with $a \geq b$, or it might be the set of pairs with $a = b$, or the set of pairs with a dividing b, etc., etc.

If $A = B$, a typical occurrence, we talk of a relation *on* A.

If $f : A \to B$ is a function, the graph of f is a relation: it's the set of pairs $(a, f(a))$.

Relations on a set A can be:

• *Transitive.* This means that if (a, b) and (b, c) are in the relation, so is (a, c). An example: divisibility among integers.

• *Symmetric.* This means that if (a, b) is in the relation, so is (b, a). Example: "being acquainted" on the set of students in this class.

• *Antisymmetric.* This means that if (a, b) is in the relation and $a \neq b$, then (b, a) isn't in the relation. Divisibility is again a fine example. The relation "equality" (on some set) is both symmetric and antisymmetric.

• *Reflexive.* This means that every pair (a, a) is in the relation. Most relations that come up in practice have this property. One that doesn't (on the set of integers): (a, b) is in the relation precisely when a is bigger than b.

Relations which are reflexive, transitive and symmetric are called equivalence relations. They are so important that they get a whole lecture to themselves (Thursday).

You can compose two relations the way you compose functions. If R is a relation from A to B and S a relation from B to C, then the composite $S \circ R$ is the set of pairs (a, c) which can be bridged in the following way: there's a $b \in B$ such that (a, b) is in R and (b, c) is in S.

If R is a relation on A, then $R \circ R$ is also called R^2. You can define R^3, R^4, etc. in a similar way.

For example, let A be the set of people at UC Berkeley, and let R be the relation of being acquainted. (Say that every $a \in A$ is self-acquainted.) Then two people are related under R^n if you can make a chain of n people so that successive people know each other. Some of the links in the chain might be trivial—two people who know each other are deemed to be related under R^2. I'm related to myself under R^9 even though I don't know anybody.

You can have n-ary relations where $n > 2$. This means a relation on $A_1, \ldots, A_n$, where the A_i are sets. By definition, such a relation is just a subset of $A_1 \times \cdots \times A_n$.

A_1 might be a set of vintage years, A_2 might be the set of wineries in Richmond, and A_3 might be the set of California wine types (cabernet, sauvignon blanc, zinfandel, ...). Then R could be the set of triples corresponding to the wines that were actually made. If Brandborg Cellars made a pinot noir in 1993, then the triple (1993, Brandborg, pinot noir) would be in the relation.

In Math 55, we won't have all that much to say about relations other than binary relations. If you want to build databases, you obviously have to learn about n-ary relations.

Binary relations are often represented by 0-1 matrices. Suppose that the sets A and B are finite, and order them. If $A = B$, just order it once. The matrix corresponding to $R \subseteq A \times B$ has columns indexed by the elements of B and rows by the elements of A. You put a 1 in the ijth place if a_i is related to b_j, and a 0 if not.

Take $A = B = \{1, 2, 3, 4, 5, 6\}$ with the obvious ordering and let R be the relation of divisibility: a is related to b if $a|b$. Here's the 6×6 matrix which I think represents it:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

One reason for writing down matrices of relations is that you can tell quickly from the matrix whether a relation is symmetric, antisymmetric, or reflexive.

A better reason is that you can compute the composite of two relations very quickly in terms of their matrices. If R is a relation from A to B and R' is a relation from B to C, then $R' \circ R$ is represented by the *boolean product* $M \odot M'$ of the matrices M and M' which represent R and R'. *Note the change in order!* This product is defined in the usual way **except** that when you do the matrix multiplication you use $\wedge$ to do the multiplications and $\vee$ to do the sums which follow the multiplications.

For example, let A be the set $\{1, 2, 3\}$, and let $B = C$ be the set $\{1, 2, 3, 4, 5\}$. Let R be the relation such that $(a, b) \in R$ if and only if $b = 2a$. Then the matrix of R is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let R' be the following relation: $(b, b') \in R'$ when $b' = b - 1$. The matrix looks like

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

To compute $R' \circ R$, you need to figure out

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

As it happens, the boolean product is also the usual product of the two matrices

It is also possible and fruitful to represent relations by directed graphs ("digraphs"). If A is a finite set, you would represent a relation R on A as follows: make a vertex for each $a \in A$ and lay down a little directed arrow (or edge) from a to a' whenever $(a, a') \in R$. We will draw a few in class.