These are the slides for November 25, 1997.

Planar Graphs

I began the slides for the previous lecture by writing

> We come to one of the most interesting topics in the latter third of Math 55.

The same statement applies today. My plan(e) is to discuss planar graphs today and to come back to Dijkstra's algorithm next week, if at all.

A graph is called *planar* if it can be drawn in the plane without crossings. A graph may be very complicated and look as though no one could draw it in the plane; nonetheless, sometimes a patient experimenter will succeed in rendering it in the plane.

Suppose that you try and try and try, but fail repeatedly to draw a graph in the plane. How could you ever *prove* that the graph is non-planar?

Euler, again

Euler's formula relates the number of edges and vertices of a graph with the number of regions that the graph divides the plane into[1]. In the formula, the entire "outside" of the graph counts as one region. The other regions are the bounded ones.

Here's the formula:

$$r = e - v + 2,$$

where r is the number of regions, e the number of edges and v the number of vertices. The formula applies to planar, simple connected graphs.

It's proved by induction: you view a graph as being built up from a very simple graph by the successive addition of edges.

For example, take C_n ($n \geq 3$). Then $r = 2$, $e = n$, $v = n$, so the formula works out. Experiment with your favorite graphs.

How to show that something is non-planar. There's a Corollary to Euler's formula: *For a connected, simple planar graph with $v \geq 3$ vertices, we have $e \leq 3v - 6$.* This is the right type of information because graphs with lots of edges are those that are apt not to be planar.

To deduce the corollary from the theorem, you define the concept of the *degree* of a region: it's the number of edges that touch the region. An edge

[1] Never end a sentence with a preposition.

that touches only one region counts double for that region. Hence the sum of the degrees of the regions is twice the number of edges; i.e., it's also the sum of the degrees of the vertices.

We'll admit that the degree of each region is at least 3. (Only two edges can't make a bounded region. There are enough vertices (≥ 3) to make the unbounded region have degree > 2.) Hence we have

$$2e = \sum \text{the degrees} \geq 3r,$$

so $(2/3)e \geq r$. Plug in $r = e - v + 2$ to get $e - v + 2 \leq (2/3)e$. Multiply by 3 and collect terms: $e - 3v + 6 \leq 0$, so $e \leq 3v - 6$.

The example given in the book is the complete graph K_n. It has $v = n$ and $e = n(n-1)/2$. The inequality $n^2 - n \leq 6n - 12$ is *not* satisfied for $n \geq 5$. Hence K_n is non-planar for $n > 4$.

We'll see in a moment that the graph $K_{3,3}$ is also non-planar. Since K_5 is non-planar, it follows that a graph is non-planar if it contains one of the two "bad graphs" K_5, $K_{3,3}$. A theorem of Kuratowski provides a converse: if a graph is non-planar, then it contains a subgraph which is *homeomorphic* to one of the two bad graphs. Don't be scared of "homeomorphic"—it is there because two graphs are essentially the same if you get from one to another by dividing an edge into two ("elementary subdivision").

How to show that $K_{3,3}$ is non-planar? It has 6 vertices and 9 edges; note that 9 is in fact less than $18 - 6$. So we need a sharper criterion. And here it is: *If a connected planar simple graph has $v \geq 3$ and no circuits of length 3, then $e \leq 2v - 4$.* Notice that $2v - 4 = 8$ in the case of $K_{3,3}$, so we'd get from this criterion that $K_{3,3}$ is non-planar. Also, it's clear that there are no circuits of odd length because the graph $K_{3,3}$ is bipartite.

How to prove the sharper criterion? The point is that the degree of each region has to be at least 4, since a region of degree 3 would lead to a circuit of length 3. Hence we get

$$2e = \sum \text{the degrees} \geq 4r,$$

so $(1/2)e \geq r$. Plug in $r = e - v + 2$ to get $e - v + 2 \leq (1/2)e$. Multiply by 2 and collect terms: $e - 2v + 4 \leq 0$, so $e \leq 2v - 4$, as required.