Announcements:

• The second midterm (Q & A) from Spring '96 is on the Web page and at Copy Central. You can also find on the Web page some review questions and a list of poker hands and frequencies.

• For the exam on October 30, the class will be divided as before. *Students who went to 1 Leconte last time will go to 10 Evans this time.* Students who were in Haas last time are again in Haas.

• The exam next Thursday will cover material through the lecture on Tuesday, October 21. It is cumulative (covers old material) but stresses material since the first midterm.

Let f be a random variable on a sample space S. Assume that the values of f are non-negative integers. The generating function of f is the series

$$\sum_{k=0}^{\infty} P(f=k)z^k.$$

For example, let S be the set of configurations of two dice, and let f be the "sum of the two dice" random variable. The generating series of f is the polynomial

$$\frac{1}{36}(z^2+2z^3+3z^4+4z^5+5z^6+6z^7 \\ +5z^8+4z^9+3z^{10}+2z^{11}+z^{12}).$$

If you think about how the coefficients were generated, you realize that this polynomial is nothing but the *square* of the corresponding polynomial for one die:

$$\frac{1}{6}(z+z^2+z^3+z^4+z^5+z^6).$$

More generally, suppose you have two not necessarily fair dice, where the probabilities of getting 1–6 are respectively $a_1,\ldots,a_6$ and $b_1,\ldots,b_6$. Then the generating function for the function f becomes

$$(a_1z+\cdots+a_6z^6)(b_1z+\cdots+b_6z^6).$$

Is it ever possible to weight the two dice so that the probabilities for 2–12 in the roll of the two dice become equal?

We are asking whether

$$(a_1z+\cdots+a_6z^6)(b_1z+\cdots+b_6z^6)$$

can ever be $\frac{1}{11}(z^2+z^3+\cdots+z^{12})$.

The answer is "no," meaning that we can never slant the dice game so that the probabilities of 2–12 are equal. This is fairly easy to see with the generating functions, but probably impossible to see otherwise.

(This whole thing was explained to me one day during the summer by a GSI who was teaching Math S55.)

The proof is by contradiction. Suppose that

$$\begin{aligned}&(a_1 + \cdots + a_6 z^5)(b_1 + \cdots + b_6 z^5)\\ =&\frac{1}{11}(1 + z + \cdots + z^{10}).\end{aligned}$$

By calculus, each factor on the left has a real root because it's an odd-degree polynomial. So of course the right-hand side has to have a real root. We will show that it doesn't, yielding the contradiction.

If r is a root of the right-hand side, it's a root of

$$(1 - z)(1 + z + \cdots + z^{10}).$$

The latter polynomial is $1 - z^{11}$, and its only real root is 1. Hence r must be 1. But, clearly, 1 is not a root of $(1 + z + \cdots + z^{10})$.

As Lenstra explains in his notes, generating functions are handy for doing a multitude of manipulations with random variables. Rather than talk more about generating functions, I want to explain about **Standard Deviation** and **Variance**:

The standard deviation of a random variable f is defined by

$$\sigma(f) = \sqrt{\sum_{x \in S} (f(x) - E(f)^2 P(x)}.$$

Thus $\sigma(f)$ is the square root of the expected value of the random variable $(f - E(f))^2$. It measures how far f varies from its mean value $E(f)$. The *variance* of f is just the square of the standard deviation:

$$V(f) = \sigma(f)^2.$$

An alternative formula for $V(f)$ is the following one:

$$V(f) = E(f^2) - E(f)^2.$$

To see this, we expand out the definition of $V(f)$, getting

$$\begin{aligned}&\sum_{x \in S} f(x)^2 P(x) - 2E(f) \sum_{x \in S} f(x)P(x)\\ &+ E(f)^2 \sum_{x \in S} P(x).\end{aligned}$$

The first term is $E(f^2)$, the second term is $-2E(f)^2$, and the third is $E(f)^2$.

Example: let S be the sample space $\{0, 1\}$ where the probability of 1 is p. Let f be the random variable taking 0 to $0 \in \mathbf{R}$ and 1 to $1 \in \mathbf{R}$. Then $E(f) = p$, so $V(f) = p - p^2 = pq$. Thus $\sigma(f) = \sqrt{pq}$.

Another example: consider the roll of two fair dice, and let t be the function "twice the first die." Then $E(t) = 7$. Also

$$\begin{aligned} E(t^2) &= \frac{1}{6}(2^2 + 4^2 + 6^2 + 8^2 + 10^2 + 12^2) \\ &= \frac{364}{6} = \frac{182}{3}. \end{aligned}$$

Hence $V(t) = 182/3 - 49 = 35/3$, so $\sigma(t) \approx 3.41565$. If you used $t/2$ instead of t, the variance would become $35/12$.

Now let f be the function "sum of the two dice." Its expected value is 7 (like that of t), but its variance smaller. Namely, f is the sum of the two functions "first die" and "second die," each with variance $35/12$. It will follow that the variance of f is $2 \cdot 35/12 = 35/6$. This is half the variance of t.

The point is that the variance of a sum is the sum of the variances when the random variables are independent. To see this, use the key relation $E(fg) = E(f)E(g)$ which was mentioned last time (but not derived out—see the notes for that). If f and g are independent (like the rolls of two different dice...), we have

$$\begin{aligned} V(f+g) &= E\left((f+g)^2\right) - E(f+g)^2 \\ &= \left(E(f^2) + 2E(f)E(g) + E(g^2)\right) \\ &\quad - \left(E(f)^2 + 2E(f)E(g) + E(g)^2\right) \\ &= V(f) + V(g). \end{aligned}$$