These are the slides for November 20, 1997.

Euler paths and Euler circuits

We come to one of the most interesting topics in the latter third of Math 55. The topic is usually introduced by a story about the seven bridges of Königsberg: [picture] Is it possible to start somewhere, cross each bridge exactly once, and return to the starting point?

The Swiss mathematician L. Euler saw that the problem was equivalent to a question about the undirected multigraph G [picture here] which summarizes the configuration of bridges and land masses: is it possible to find a simple circuit which contains every edge of G? Such a simple circuit is now called an *Euler circuit*.

An *Euler path* in a graph G is a simple path which contains every edge of G.

Suppose that G has an Euler circuit. Then it's easy to see that every vertex has even degree. Here's why: Imagine an Euler circuit on G. Take a vertex v which is not the starting point. Each time that the circuit enters v, it leaves v by a different edge. Thus, the visit to v accounts for a contribution of "2" to the degree of v. Since every edge is used by the circuit, we get the full degree of v in the count. It's easy to deal with that v which is the starting point (e.g., by changing the starting point).

The **Theorem** of Euler is that the converse holds: *If G is a connected multigraph in which every vertex has even degree, then G has an Euler circuit.*

How to see this? Start with an arbitrary vertex; call it x_0. Choose an edge going out of x_0; say it hits x_1. Keep going as long as possible: make the longest simple path that you can (given your successive choices). At some vertex, you get stuck. That vertex must be x_0 because of the parity — each time you enter a vertex different from x_0, you can find another vertex that will get you out. Example:

To do better, consider the graph H gotten by retaining the edges which have not yet been used and the vertices that touch those edges: [picture]

The graph H still has the "even degree" property, but it might not be connected.

The graph H has a vertex in common with the graph that was erased to make H. Call this vertex v. Make a simple path which starts at v and is as long as possible. It's a circuit. This circuit and the previously-constructed circuit can be spliced together to make a bigger circuit. Continue until there's nothing left!

A similar theorem: *A connected multigraph G has an Euler path but not an Euler circuit if and only if the number of vertices of odd degree is 2.*

A *Hamiltonian path* in a graph G is a path that visits each vertex exactly once. A *Hamiltonian circuit* on G is a circuit that visits each vertex exactly once, except that the starting vertex is also the ending vertex. It's possible that not all edges will be used in such paths!

A silly example occurs with K_n, the complete graph on n vertices. Just take any path from vertex 1 to vertex 2,..., until you get to the last vertex. You have a Hamiltonian path. To get a Hamiltonian circuit, return to vertex #1.

Amazingly, there's no simple criterion to decide whether a given graph has a Hamiltonian path or circuit.

Weighted Graphs

We come to a new topic, associated with graphs whose edges have "weight" assigned to them. The weight can be thought of the cost (in some appropriate sense) of traversing the edge. The fundamental problem is as follows: find the cheapest path from a vertex v to a vertex w (assuming that the graph is connected). One answer is given by Dijktra's algorithm, which we will now learn.

Example: find the cheapest path from v to w:

The technique is to find the closest vertex to v (taking into account the weights), then the second closest vertex, and so on. When w has gotten listed, stop.

In the picture, a is clearly the closest vertex to v: its distance from v (in the sense of total cost or weight) is 1. The second closest is reached either directly from v or via a: it's clearly b, at distance 2 (reached in either of two ways). The next closest is c, at distance 4. Then either e or f (tie), at distance 5. Then d, at distance 6. Then you can get to w, at distance 11.

On Tuesday, I will explain more clearly how this algorithm is supposed to work. Then we will go on to the subject of planar graphs. On December 2, I will begin by finishing up §7.7, and then begin taking questions related to the final exam. Our last class session is December 4.