A bunch of announcements:

• The second midterm (Q & A) from Spring '96 is on the Web page and at Copy Central.

• For the exam on October 30, the class will be divided as before. *Students who went to 1 Leconte last time will go to 10 Evans this time.* Students who were in Haas last time are again in Haas.

• I posted the homework for the Lenstra Notes. Homework due next Wednesday: the first five questions in those notes.

For the Lenstra notes, the sample space S can be countably infinite, but it's usually finite as for Rosen. Why only countably infinite? Because this is a course in *Discrete* Math.

The space S comes equipped with a probability function $p : S \to [0,1]$ such that $\sum_{x \in S} p(x) = 1$. Note that the sum might be an infinite series.

Example: $S = \{1, 2, 3, \dots\}$; $p(n) = 2^{-n}$.

Lenstra writes P instead of p and calls his events A and B instead of E and F. Can we cope??

For $E \subset S$ an event, we define $p(E) = \sum_{x \in E} p(x)$. This is what we've had before.

Conditional probability: we put $p(E|F) = \dfrac{p(E \cap F)}{p(F)}$ when the denominator is $\neq 0$. If $p(F) = 0$, we punt.

Here is a conditional probability question, sort of: "Pokerhontas drops the King of $\clubsuit$, the King of $\heartsuit$ and the Ace of $\spadesuit$ into a large bag. She shakes the bag vigorously and then removes two of the cards without looking at them. (a) If it is known that one of the two cards is a king, what is the probability that both cards are kings? (b) If it is known that one of the two cards is the King of $\heartsuit$, what is the probability that both cards are kings?"

This was on my final last time around. I got a lot of appreciative chuckles during the exam.

Answer to (a): Since she certainly removes at least one king, the "If it is known that…" gives no information! In symbols, $p(F) = 1$ and $E \cap F = E$. What is the probability of E, the event that she pulls out the two kings? This is clearly 1/3.

Answer to (b): There are precisely two sets of two cards that have the King of $\heartsuit$ in them. Here F has 2 elements. One of these sets has the other king. Hence the answer is 1/2.

Independence: E and F are independent if $p(E \cap F) = p(E)p(F)$. When $p(F) \neq 0$, this means $p(E|F) = p(E)$, i.e., that F "doesn't affect" E.

Example from Lenstra notes: consider the roll of two dice. Let A be the event "the first die is 3"; let B be "the total is 7"; let C be "the total is 3." Of the three possible pairs of events, which ones are independent pairs?

Random variables are functions $f : S \to \mathbf{R}$; that's all there is to the definition. Imagine S representing the various configurations of some gambling device (roulette wheel, say) and $f(s)$ representing the payoff when it's s that turns off.

Let $S = \{ (a, b) \,|\, 1 \leq a, b \leq 6 \}$ be the set of configurations of two dice. We might take $f(a, b) = a + b$. Other examples are $(a, b) \mapsto a$ and $(a, b) \mapsto b$.

If f is a random variable and r is a real number, the "probability that f is r" is

$$p(f = r) := \sum_{x : f(x) = r} p(x).$$

It's $p(E)$ where E is the event "$f = r$." The *expected value* of f is

$$E(f) := \sum_{x \in S} f(x)p(x) = \sum_{r} p(f = r) \cdot r.$$

In the latter sum, it suffices to sum over those real numbers r that are values of f, so the sum is either finite or an infinite series.

A first example: start with an event A and let $f = \chi_A$ be the characteristic function of A. Then $E(\chi_A) = p(A)$.

Next, let S be the set of pairs (a, b) as above. Each pair has probability $1/36$ attached to it. What is the expected value of the function $f : (a, b) \mapsto a + b$? There is an easy way and a hard way to get the answer. I'll do both.