These are the notes for November 11, 1997.

Our topic today is "partial orderings." A partial order (or ordering) on a set S is an order—i.e., a subset R of $S \times S$—which is reflexive, transitive and anti-symmetric.

Recall that an equivalence relation on S is a relation which is reflexive, transitive and symmetric. Thus the only new element is the "anti-" in the definition. However, the whole subject has a completely different flavor from the topic of Thursday's lecture!

Notation: one writes $a \preceq b$ if $(a, b) \in R$ and $a \prec b$ if $a \preceq b$ and one knows that a and b are distinct.

Here are some basic examples:

(1) The relation of inclusion on subsets of a set. If $S = \wp(A)$, then elements of S are subsets of A. If B and C are two such subsets, we write $B \preceq C$ if $B \subseteq C$. Notice that two random elements of S will probably not be "comparable."

(2) The relation of divisibility on the set of positive integers: $n \preceq m$ means that n divides m.

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(3) The relation "$\leq$" on the set of positive integers.

This third example has a special property: any two elements of the set are comparable. When this happens, we say that the set has been "totally ordered."

(4) Lexicographic orders.... Given partial orders on several sets $S_1, S_2, \ldots$, we can make a partial order on their product $S_1 \times S_2 \times \cdots$. For instance, suppose that we have partial orderings on S_1 and S_2. We can define $(s_1, s_2) \prec (s_1', s_2')$ to mean that either $s_1 \prec s_1'$ or that $s_1 = s_1'$ and that $s_2 \prec s_2'$. This construction is especially popular in the case of total orderings.

Here is a variant of this idea for strings. Take S to be a partially ordered set and consider strings $a_1 a_2 \cdots a_m$ and $b_1 b_2 \cdots b_n$ made out of elements of S. We define

$$a_1 a_2 \cdots a_m \prec b_1 b_2 \cdots b_n$$

in the following way. Let t be the smaller of n and m. If $(a_1, \ldots, a_t) \prec (b_1, \ldots, b_t)$, then $a_1 a_2 \cdots a_m \prec b_1 b_2 \cdots b_n$. If $(a_1, \ldots, a_t) = (b_1, \ldots, b_t)$, then $a_1 a_2 \cdots a_m \prec b_1 b_2 \cdots b_n$ if it happens that the second string is longer (i.e., $n > m$).

This is more or less the way we compare words in a dictionary. Note that "lexico" refers to "lexicons" or dictionaries.

We can represent partial orderings using directed graphs as at the end of Thursday's lecture. However, it has proved convenient to use "Hasse diagrams" instead. These are just the directed graphs in which redundant information has been removed. Here's an example:

$$\begin{tikzcd}
 & & 14 & & 15 \\
 & & & 13 \arrow[ul] \arrow[ur] & \\
6 & & 12 \arrow[ur] & & 11 \arrow[ul] \\
 & 5 \arrow[ul] \arrow[ur] & & & 9 \arrow[u] \\
 & 4 \arrow[u] & 10 \arrow[uu] & & 8 \arrow[u] \\
 & 3 \arrow[u] & & 7 \arrow[ul] \arrow[ur] & \\
 & 2 \arrow[u] & & & \\
 & 1 \arrow[u] & & &
\end{tikzcd}$$

FWIW, this diagram appears (somewhat differently) on one of the first pages of a number theory book, "Local Fields" by J-P. Serre. The intent is to indicate logical relations among chapters. For example, you need to have read chapters 1–5 before opening up chapter 6, and reading chapter 12 requires familiarity with the contents of the first five chapters, as well as chapters 7 and 10. On the other hand, you can pick up the book at the beginning of chapter 7 and read 7–11 without consulting the first five chapters.

The point is that $a \prec b$ means here that a must be read before b. There can be several elements a of the set of chapters with the property that $a \prec b$.

Anecdote: I included this Hasse diagram on my final exam three semesters ago, saying only that it was from a "famous number theory book." I sent a copy of the exam to Serre in Paris. Serre wrote back to say that he had found the examination interesting and challenging, but added that he was able to do several of the problems.

Posets can have **maximal**, **minimal**, **least** and **greatest** elements. An element a is maximal if there is no $b \neq a$ with $a \preceq b$. It's the greatest element if $b \preceq a$ for all b. For "minimal" and "least elements," stand on your head. As you can imagine, it's much more common to have maximal elements running around than to have greatest elements.

Note the Lemma: *Every finite non-empty partially ordered set has a minimal element.* To prove the statement, take an element at random. If it's not minimal, choose a smaller element. If this latter element is minimal, stop; otherwise, keep going. . . . The process will end because the set is finite.

In the example from Serre's book, it looks to me that 1 and 7 are the two minimal elements and that 6, 14 and 15 are maximal.

If you take the set of positive integers under $\leq$ or divisibility, then 1 is clearly the least element (and is thus also minimal), but there are no maximal elements. If $S = \{1, 2, 3, 4, 5\}$ under divisibility, then the maximal elements are 3, 4 and 5.

There's an amazing theorem in this section of the book. The buzz phrase: "topological sorting." The statement is as follows. Suppose that you have a finite partially ordered set (S, R). Then there's a total ordering $\preceq$ on S so that $a \prec b$ whenever $a \ R \ b$. In words: every partial ordering can be refined into a total ordering. Or: there is a total ordering which is compatible with every partial ordering.

Here's the proof. Assume that the poset S is non-empty; otherwise, there's nothing to do. Take a minimal element a_1 in S. Then take a minimal element a_2 in $S - \{a_1\}$. Then take a minimal element a_3 in $S - \{a_1, a_2\}$. Continue until there are no more elements in S. The total ordering defined by $a_1 \prec a_2 \prec a_3 \prec \cdots$ will work.

In the example of Serre's book, a total ordering which is compatible with the given partial ordering is

$$1 \prec 2 \prec 3 \prec \cdots \prec 15.$$

Namely, you can satisfy the author's requirements by reading the chapters of the book in their usual order.

Here's another possible order:

$$1, 7, 2, 3, 10, 4, 8, 5, 12, 6, 9, 12, 11, 13, 15, 14.$$