Math 250A, Fall 2004
First Midterm Exam—September 30, 2004

This exam looks easy to me in the sense that most students are getting most of the points on most of the problems. I'm writing this page incrementally as I grade the problems. You'll see from the grade distribution how people actually did. These are Ribet-written solutions. Chu-Wee sent me his write-up of solutions write after the exam, and I thank him for his help.

1. *If n is a positive integer, find an $m \geq 1$ so that the alternating group $\mathbf{A}_m$ contains a subgroup isomorphic to the symmetric group $\mathbf{S}_n$.*

This is a fairly standard problem that I've seen in books. When I first encountered it, I thought immediately that one should take $m = 2n$. To each permutation $\sigma \in \mathbf{S}_n$, one associates the permutation that replicates σ on $\{1, \ldots, n\}$ and then repeats σ on the second block $\{n+1, \ldots, 2n\}$, sending $n+i$ to $n+\sigma(i)$ for $i = 1, \ldots, n$. This latter permutation factors as a product of $2d$ transpositions if σ factors as a product of d transpositions. Hence the associated permutation on $2n$ letters is even.

In his write-up of solutions, Chu-Wee had the same idea that I did. The majority of the students seemed to prefer another way of doing the problem, however: they took $m = n+2$. One associates to σ the permutation on $n+2$ letters that is σ itself when σ is even and the product $\sigma(n+1\, n+2)$ when σ is odd. We get in this way an element of $\mathbf{A}_{n+2}$ for each σ. Students checked easily that this association is an injective homomorphism $\mathbf{S}_n \hookrightarrow \mathbf{A}_{n+2}$, as required.

2. *Prove that every group of order* $312 = 2^3 \cdot 3 \cdot 13$ *has a proper non-trivial normal subgroup.*

The number of 13-Sylow subgroups divides $312/13 = 24$ and is congruent to 1 mod 13. Hence it's 1. There is thus a unique 13-Sylow subgroup P; P must be normal because, for each g in the group, gPg^{-1} is a 13-Sylow subgroup and thus must be equal to P. Note: the theorem that all p-Sylow subgroups of a group are conjugate is not used in this argument. It's irrelevant.

3. *Let G be a group of order 120, and let $H \subseteq G$ be a subgroup of order 24. Suppose that there is at least one left coset of H in G (other than H itself) that is also a right coset of H in G. Prove that H is a normal subgroup of G.*

Suppose that $xH = Hy$. Since x is in Hy, we have $Hx = Hy$, so $xH = Hx$, which means that x normalizes H. The hypothesis is that there is an x not in H that has this property. Hence the normalizer of H is strictly bigger than H. Since $(G : H) = 5$ is a prime number, the normalizer must be all of G.

4. *Let G be a group and let H be a subgroup of G such that the index $(G : H)$ is finite. Prove that there is a normal subgroup H_0 of G such that $H_0 \subseteq H$ and such that $(G : H_0)$ is finite. Show further that there is an $n \geq 1$ so that $g^n \in H$ for all $g \in G$.*

We have seen in homework that there is an H_0 with the indicated property. Since I asked you to prove that there is a group like this, I won't give credit to students who assert simply that they already did the problem on homework; what's required is that you actually give the argument. Once you have the H_0, you finish off the problem as follows: If $n = (G : H_0)$, then $g^n \in H_0$ for all $g \in G$. In particular, $g^n \in H$ for all $g \in G$. Note that one cannot take $n = (G : H)$. For example, let $G = \mathbf{S}_3$ and let H be a subgroup of order 2 in G. Then it is not true that $\sigma^3 \in H$ for all $\sigma \in G$. Indeed, if σ is one of the transpositions that are not in H, $\sigma^3 = \sigma$ will not be in H.

5. *Let G be a finite group, and let H be a normal subgroup of G. Let P be a p-Sylow subgroup of H, and let K be the normalizer of P in G. Establish the equality $G = HK$.*

For $g \in G$, consider gPg^{-1}. It's still in H because H is normal, and it has the same size as P. Hence it's a p-Sylow subgroup of H, so it may be written as hPh^{-1} for some $h \in H$. The equation $gPg^{-1} = hPh^{-1}$ gives that $h^{-1}g$ normalizes P. Thus we may write $h^{-1}g = k$ with $k \in K$. Then $g = hk$ yields that g is in HK.