Final Exam **Math 250A** **December 14, 1992**

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1 (6 pts.) Show that every group of order 56 has a proper normal subgroup other than $\{e\}$.

2 (7 pts.) Let G be a finite group, and let S be a finite G-set. For each $g \in G$, let $f(g)$ be the number of fixed points of g, i.e., the number of s such that $gs = s$.

(3 pts.) a. For each $s \in S$, show that $\displaystyle\sum_{t \in \mathrm{Orbit}(s)} \frac{1}{\#(\mathrm{Orbit}(t))} = 1$.

(4 pts.) b. Show that the number of orbits of G in S is the quantity $\displaystyle\frac{1}{\#(G)} \sum_{g \in G} f(g)$.

3 (6 pts.) Let A be a finite abelian group with the following property: for each $n \geq 1$, the group $\{\, a \in A \,|\, na = 0 \,\}$ has at most n elements. Prove that A is cyclic. [Use the basic structure theorem for abelian groups, or argue directly.]

4 (10 pts.) Let V be a vector space over the field k. Let $R = \operatorname{End} V$ be the ring of k-linear transformations $V \to V$.

(3 pts.) a. Suppose that W is a k-vector space. Let $M = \operatorname{Hom}(W, V)$ be the space of k-linear transformations from W to V. Show that the operation $r\colon m \mapsto r \circ m$ makes M into a left R-module.

(4 pts.) b. If W is isomorphic to a direct sum of finitely many copies of V, show that M is free over R.

(3 pts.) c. Give an example where M is simultaneously free of rank 1 over R and free of rank 2 over R.

5 (12 pts.) Suppose that E is an algebraic extension of K.

(4 pts.) a. Let $f(x)$ be a non-zero polynomial with coefficients in E. Show that there is a non-zero polynomial $g(x) \in K[x]$ such that $f(x)$ divides $g(x)$ in $E[x]$.

(4 pts.) b. Suppose that E contains a splitting field for each non-constant polynomial with coefficients in K. Prove that E is algebraically closed.

(4 pts.) c. Suppose that K has characteristic 0. Assume that every non-constant polynomial over K has a root in E. Prove that E is algebraically closed. [Use the Primitive Element Theorem.]

6 (9 pts.) Let R be a ring, and let M, P and Q be left R-modules. Suppose that $\alpha\colon P \to M$ and $\beta\colon Q \to M$ are each surjective R-module maps. Form

$$A = \{\, (x, y) \in P \oplus Q \,|\, \alpha(x) = \beta(y) \,\}.$$

(5 pts.) a. Establish an exact sequence $0 \to \ker\beta \to A \to P \to 0$.

(4 pts.) b. If P and Q are projective, prove that $P \oplus \ker\beta$ and $Q \oplus \ker\alpha$ are isomorphic.
