Math 115 Professor K. A. Ribet

Midterm Exam September 27, 2006

This exam was an 50-minute exam. It began at 2:10PM. There were 3 problems, for which the point counts were 9, 10 and 11. The maximum possible score was 30.

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Your paper is your ambassador when it is graded. Correct answers without appropriate supporting work will be regarded with extreme skepticism. Incorrect answers without appropriate supporting work will receive no partial credit. This exam has five pages, including this cover sheet and a blank page at the end. Please write your name on each page. At the conclusion of the exam, please hand in your paper at the front of the room.

1. *Label the following statements as TRUE or FALSE, giving a short explanation (e.g., a proof or counterexample) for your answer:*

a. *If an integer ≥ 2 divides the product of two positive integers, then it cannot be relatively prime to both of the integers.*

This is true. If a divides bc and is prime to c, then a divides b. Thus a is a common divisor of a and b. If a is greater than 1, a and b are thus not relatively prime.

b. $\operatorname{ord}_p(n+m) = \min\{\operatorname{ord}_p n, \operatorname{ord}_p m\}$ *for $p \geq 5$ and positive integers n and m. [When n is a positive integer and p is a prime, recall that $\operatorname{ord}_p n = t$ if p^t is the highest power of p dividing n.]*

This looks vaguely attractive, but it's false. For example, $\operatorname{ord}_5(15) = \operatorname{ord}_5(10) = 1$, but $\operatorname{ord}_5(15+10) = \operatorname{ord}_5(25) = 2$.

c. *If a product of an even number of prime numbers is 1 plus a multiple of 4, the product is a sum of two perfect integer squares.*

If p and q are distinct primes that are 3 mod 4, their product is 1 mod 4 but isn't a sum of two squares (as we discussed in class). For example, $21 = 3 \cdot 7$ is not a sum of two squares. Of course, you can check this directly by listing the squares less than 21 and observing that no two of them add up to 21.

2. *Suppose that p is a prime number such that $2^p - 1$ is prime. Show that the sum of the positive divisors of $2^{p-1}(2^p-1)$ is $2^p(2^p-1)$. [Only 44 such prime numbers p are known. The most recently discovered such prime is 32582657; according to* `http://primes.utm.edu/`*, mathematicians at the University of Central Missouri discovered on September 6 that $2^{32582657} - 1$ is prime.]*

Let $q = 2^p - 1$, so q is prime by hypothesis. The divisors of $2^{p-1}q$ are the numbers 2^i with $0 \le i \le p-1$ and the numbers $2^i q$ with i in the same range. Adding up these numbers, we get $(1+2+4+\cdots+2^{p-1})(1+q)$. The sum of the powers of 2 is $2^p - 1$ (geometric series formula, or otherwise) and the number $1+q$ is 2^p. This gives what is wanted.

A number whose divisors sum to twice the number is called *perfect*. It is fairly easy to show that all even perfect numbers are of the form $2^{p-1}(2^p-1)$ with $2^p - 1$ prime; the first two such numbers are 6 and 28. It is widely expected that there are no odd perfect numbers, but this statement has not been proved (so far).

3. *When n is a positive integer, show that the prime factorization of $\binom{3n}{n}$ involves no primes p with $3n/4 < p \le n$ and no primes p with $3n/2 < p \le 2n$. For example, if $n = 31$, then $3n/4 = 23.25$ and $3n/2 = 46.5$. In the factorization of*

$$\binom{93}{31} = 2^5 \cdot 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 37 \cdot 41 \cdot 43 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot 89,$$

the primes 29 and 31 do not appear and the primes 47, 53, 59 and 61 are absent as well.

In class, I added the hypothesis that n is at least 5. For $n = 3$, we have $\binom{9}{3} = 84$, which is divisible by n, so the problem would be false without some extra hypothesis. (Thanks to the student who pointed this out during the exam!) The point is that we are dealing only with primes that are greater than $3n/4$ and we want their squares to be greater than $2n$. For this, we need $9n^2/16 \ge 2n$, or $n \ge \dfrac{32}{9}$. The hypothesis $n \ge 5$ certainly ensures this!

We have a formula for $\operatorname{ord}_p \binom{3n}{n}$, namely $\displaystyle\sum_{k=1}^{r} \left(\lfloor \frac{3n}{p^k} \rfloor - \lfloor \frac{2n}{p^k} \rfloor - \lfloor \frac{n}{p^k} \rfloor \right)$, where r can be taken to be any integer such that $p^{r+1} > 2n$. As explained in the previous paragraph, we can take $r = 1$, so that

$$\operatorname{ord}_p \binom{3n}{n} = \lfloor \frac{3n}{p} \rfloor - \lfloor \frac{2n}{p} \rfloor - \lfloor \frac{n}{p} \rfloor.$$

To say that p does not occur in the factorization of $\binom{3n}{n}$ is to say that $\operatorname{ord}_p \binom{3n}{n} = 0$. This means that we have to establish the formula $\lfloor \frac{3n}{p} \rfloor = \lfloor \frac{2n}{p} \rfloor + \lfloor \frac{n}{p} \rfloor$ when p is in the two ranges $3n/4 < p \le n$ and $3n/2 < p \le 2n$. In the first range, we have $1 \le n/p < 4/3$, so that $2 \le 2n/p < 8/3$ and $3 \le 3n < 4$. Thus $\lfloor n/p \rfloor = 1$, $\lfloor 2n/p \rfloor = 2$ and $\lfloor 3n/p \rfloor = 3$. Since $3 = 2 + 1$, we are OK. In the second range, $1/2 \le n/p < 2/3$, $1 \le 2n/p < 4/3$ and $3/2 \le n/p < 2$; the equation $\lfloor \frac{3n}{p} \rfloor \stackrel{?}{=} \lfloor \frac{2n}{p} \rfloor + \lfloor \frac{n}{p} \rfloor$ is true because $1 = 1 + 0$.