Discrete Mathematics

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Everyone enrolled in the university has lived in a dormitory.
Mia has never lived in a dormitory.

Mia is not enrolled in the university.

A convertible car is fun to drive.
Fred's car is not a convertible.

Fred's car is not fun to drive.

Odd integers are those of the form $2k + 1$. Even integers are those of the form $2k$.

Theorem

If n is an odd number, then n^2 is odd.

Theorem

The product of two odd numbers is odd.

Theorem

If n is an odd number, then n^2 is of the form $4k + 1$.

Theorem

If n is even, n^2 is of the form $4k$.

Rosen, top of p. 83: “Note that every integer is even or odd, and no integer is both even and odd.”

Can we prove these assertions?

Theorem

Let d be a positive integer and let n be an integer. Then there are unique integers q and r such that $n = qd + r$ and $0 \leq r < d$.

This theorem can be proved easily by *mathematical induction* (§5.1). The case $d = 2$ corresponds to the assertion that every integer is either even or odd.

Proof by contraposition

This means: proof by passing to the contrapositive.

Theorem

If n^2 is odd, n is odd.

We can prove this as follows (except that the slide needs to be corrected):

$$\begin{aligned} n \text{ not even} &\longrightarrow n \text{ odd} \\ &\longrightarrow n^2 \text{ odd} \\ &\longrightarrow n^2 \text{ not even} \\ &\longrightarrow n^2 \text{ odd.} \end{aligned}$$

Theorem

Suppose that $n = ab$, where a and b are positive integers. Then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

If the conclusion is false, a and b are both $> \sqrt{n}$ and therefore their product is greater than $\sqrt{n}\sqrt{n} = n$, contrary to the assumption $n = ab$.

This looks to me like a proof by contraposition.

Theorem

There is no rational number whose square is 2.

This theorem is often paraphrased as the statement that $\sqrt{2}$ is irrational. I prefer the statement that I've chosen because it does not make reference to any square root of 2. The fact that there is a real number whose square is 2 is actually a hard and deep fact.

The proof of the theorem begins with the assumption that there *is* a rational number whose square is 2. It ends with the observation that we have a contradiction.

In the middle of the proof, we say that we can write $2 = (a/b)^2$ where a and b are integers that are not both even. (The idea is that if a and b are both even, we can divide them both by 2 and get smaller numbers a and b. . . .) We then write $a^2 = 2b^2$, note that a^2 is even and deduce that a is even. If $a = 2k$, we get $4k^2 = 2b^2$ and then $b^2 = 2k^2$. We see that b^2 is even, so b must be even, and we have a contradiction.

Not bad, eh?

Proof by “cases” or by “exhaustion”

Theorem (c. 1990)

If n is an integer bigger than 2 and less than 10^6, there are no non-zero integers a, b and c such that

$$a^n + b^n = c^n.$$

This statement consists of 999,999 different theorems!

Perfect number: a positive integer n such that n is the sum of its divisors $< n$:

$$6 = 1 + 2 + 3;$$

$$28 = 1 + 2 + 4 + 7 + 14.$$

No odd perfect numbers are known. Even perfect numbers are easy to describe: they're of the form $2^{p-1}(2^p - 1)$ where p is a prime for which $2^p - 1$ is also a prime. The two examples above correspond to the cases $p = 2$ and $p = 3$.

Theorem

All perfect numbers less than 10^6 *are even.*

One plausible proof of the theorem is to check each integer $< 10^6$ to see whether it's a perfect number. If it is, figure out whether it's even or odd.

Theorem

Every perfect square is either a multiple of 4 or 1 more than a multiple of 4.

Theorem

Suppose that x and y are integers for which xy and $x + y$ are both even. Then x and y are both even.

A stab at a proof: The product of two odd numbers is odd, so that x and y cannot both be odd. Thus at least one is even. *Without loss of generality*, we can suppose that x is even. Then x and $x + y$ are both even, so their difference y is even as well.

Faculty Club breakfast

How many of you would join me for breakfast on Friday morning at 7:30AM in the Faculty Club? We'd meet in the sit-down dining room. There's a "special breakfast" that includes two eggs, toast, coffee, orange juice, some fruit and more. It's \$3.99 plus tax plus tip—I usually collect \$5.50 or \$6 from each student.

Show of hands?

For Many Students, Print Is Still King

This is from today's *Chronicle of Higher Education* Despite the hype about e-books, the classic textbook hasn't gone away. In fact, the hold-it-in-your-hands book remains the first choice for many instructors and students. Even as publishers scramble to produce new kinds of content for a digital learning environment, print is still king for many of the biggest-selling textbooks. Take the familiar Norton Anthology of English Literature, which celebrated its 50th anniversary this past year. Norton doesn't even offer an electronic version, but the book is going strong in its ninth edition. Its success has spawned a long line of Norton anthologies, devoted to American literature, African-American literature, children's literature, Latino literature, and more. . . .