Math 55 final exam, May 16, 2019

You acted with honesty, integrity, and respect for others.

Problem	1	2	3	4	5	6	7	8	9	Total
Points	8	6	7	8	8	8	8	8	8	69

1a. Use the equation $1 = 121 \cdot 18 - 7 \cdot 311$ to find an integer x that satisfies the congruence $18x \equiv 7 \bmod 311$. (There's no need to simplify your answer.)

b. How many solutions are there to this congruence if solutions are counted as the same if and only if they are congruent mod 311?

2. Let $\{a_n\}$ be the sequence defined by $a_0 = 0$, $a_1 = 1$,

$$a_{n+2} = 3a_{n+1} - a_n \text{ for } n \geq 0.$$

Show that a_n is the Fibonacci number f_{2n} for all $n \geq 0$.

3. A die is rolled repeatedly until two different faces have come up. Explain why the expected number of rolls is $1 + \frac{6}{5}$.

4. What is the probability that a 5-card poker hand has at least two face cards? (The face cards are the jacks, queens and kings. Thus there are 12 face cards in a standard 52-card deck.)

5. Before going on vacation for a week, you ask an unreliable friend to water your ailing plant. Without water, the plant has a 90 percent chance of dying. With water, it has a 20 percent chance of dying. The probability that your friend will forget to water it is 30 percent.

a. What is the probability that your plant will be dead at the end of the week?

b. If the plant is dead when you return, what is the probability that your friend forgot to water it?

6. The expansion of $(x+y+z)^3$ contains ten terms after like terms are collected:

$$(x+y+z)^3 = x^3 + 3x^2y + 3xy^2 + y^3 + 3x^2z + 6xyz + 3y^2z + 3xz^2 + 3yz^2 + z^3.$$

After like terms are collected, how many terms are there in the expansion of $(x+y+z+w)^{100}$?

7. Suppose that A is a finite set with at least two elements and that R is an equivalence relation on A. Show that there are two distinct elements of A whose equivalence classes under R have the same size.

8. Math 55 students Alice and Bob announce their RSA public keys as (n, a) and (n, b); because they are good friends, they use the same modulus n. Their exponents a and b are relatively prime. After learning that Charlie employed RSA to send the same message to Alice and Bob, Eve succeeds at retrieving the encrypted texts that Charlie sent to the two recipients. How can Eve recover Charlie's plain text from the two encrypted texts?

9a. For which values of n does the complete graph on n vertices have an Euler circuit?

b. If A, B and C are sets such that $A \cap C = B \cap C$ and $A \cup C = B \cup C$, does it follow that $A = B$? (Give a proof or a counterexample.)