A sample Math 54 final exam

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Problem # 1

Show that the vectors $(1, 1, 3)$, $(1, 3, 1)$, $(0, 0, 1)$, and $(0, -2, 1)$ in $\mathbf{R}^3$ are dependent. You may quote any relevant theorem.

Problem # 2

Find the inverse of the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Problem # 3

Find a basis for the nullspace of the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem # 4

Sketch the phase plane flow for the system

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{x}.$$

Be sure to mark carefully the directions of the flow lines.

You do not need to write down the exact solution.

Problem # 5

Write down the solution $u(x,t)$ of the partial differential equation $u_t = 16u_{xx}$ for $0 < x < \pi$ and $t > 0$, $u = 0$ for $x = 0, \pi$ and $t \geq 0$, $u = \sin(3x) + 3\sin(5x)$ for $0 \leq x \leq \pi$ and $t = 0$.

If you remember the form of the solution, you do not need to go through the entire process of deriving it.

Problem # 6

Apply the Gram-Schmidt procedure to the vectors

$$\mathbf{v}_1 = (1, 1, 1),\ \ \mathbf{v}_2 = (0, 1, 1),\ \ \mathbf{v}_3 = (0, 0, 1),$$

to generate an *orthonormal* set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Problem # 7

The function $u(x,t) = X(x)T(t)$ solves the PDE

$$u_t = u_{xx} + 6u_x.$$

What ODE must X and T satisfy?

Problem # 8

Let $y_1(x)$ and $y_2(x)$ be two solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0.$$

Show that the Wronskian

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

solves the ODE

$$W' + p(x)W = 0.$$

Problem # 9

Find the general solution of the system

$$\mathbf{x}' = \begin{bmatrix} -8 & -1 \\ 16 & 0 \end{bmatrix} \mathbf{x}.$$

Problem # 10

Find the determinant of the matrix

$$\begin{bmatrix} 2 & 4 & 1 & 1 & 1 \\ 4 & 2 & 0 & 7 & 7 \\ 2 & 4 & 2 & 1 & 1 \\ 2 & 4 & 1 & 2 & 1 \\ 2 & 4 & 1 & 1 & 3 \end{bmatrix}.$$

(Hint: Think first about how to simplify the structure of the matrix.)

Problem # 11

For $-\pi < x < \pi$ we can write

$$|x| = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(mx) + b_m \sin(mx).$$

Compute the Fourier coefficients $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$.

Problem # 12

We are given data points $(x_1, y_1), \ldots (x_n, y_n)$ and wish to find a line

$$y = mx + b$$

that minimizes the total *least squares error*.

Show how to set up this problem in the form

$$A\mathbf{x} = \mathbf{y} - \epsilon,$$

as discussed in class. Write down the *normal equations* for the unknowns m and b.

Problem # 13

Let A be an $m \times n$ matrix. Prove that $NS(A)$ is perpendicular to $RS(A)$.

Problem # 14

Let $\mathbf{u}$ and $\mathbf{v}$ be elements of an inner product space V.

State and prove the *Cauchy–Schwarz inequality* for $\mathbf{u}, \mathbf{v}$.

(Hint: $||\mathbf{u} - t\mathbf{v}||^2 \geq 0$.)

Problem # 15

The $n \times n$ *matrix function* $X(t)$ solves the ODE

$$X'(t) = AX(t) - X(t)A, X(0) = B.$$

Assume that λ is an eigenvalue of B with eigenvector $\mathbf{x}_0$, and that $\mathbf{x}(t)$ solves

$$\mathbf{x}'(t) = A\mathbf{x}(t), \mathbf{x}(0) = \mathbf{x}_0.$$

Prove that for each time t, λ is an eigenvalue of $X(t)$, with eigenvector $\mathbf{x}(t)$.

(Hint: Define

$$\mathbf{y}(t) = X(t)\mathbf{x}(t) - \lambda\mathbf{x}(t)$$

and show $\mathbf{y}'(t) = A\mathbf{y}(t)$.)