Math 250A, Fall 2004
Problems due September 21, 2004

1. *Let G be a group of order $p^n t$, where p is prime to t and $n \geq 1$. As in class, we let S be the set of subsets of G having p^n elements and we view S as a G-set in the natural way. (If $X \in S$, gX is the set of gx with $x \in X$.) Recall from class that the size of the orbit of X is either exactly t or else is a multiple of p. Show that every orbit of size t contains exactly one p-Sylow subgroup of G and deduce the congruence $\#(S) \equiv st \bmod p$, where s is the number of p-Sylow subgroups of G.*

If P is a p-Sylow subgroup of G, let $O(P) \subseteq S$ be the orbit of P, where we consider P as an element of S. This construction gives us a map from the set of p-Sylow subgroups of G to the set of orbits for the action of G on S. The stabilizer of $P \in S$ clearly contains P. By what we saw in our class discussion, the stabilizer must be exactly P and $O(P)$ must contain exactly t elements. We need to check that the map from the set of Sylow subgroups to the set of orbits with t elements is 1-1 and onto. If P and P' are p-Sylow subgroups such that $O(P) = O(P')$, then $P' = gP$ for some $g \in G$. Since gP contains the identity element of G, the inverse of g must be in P. Thus $g \in P$ and $P' = gP = P$. Hence the map is 1-1. To see surjectivity, suppose that the orbit of $X \in S$ has t elements. Then the stabilizer of X is a p-Sylow subgroup P. Take $x \in X$; then we get $X = Px$. The orbit of X is also the orbit of $x^{-1}X = x^{-1}Px$, so it's $O(Q)$ where Q is the p-Sylow subgroup $x^{-1}Px$.

The indicated congruence follows immediately because S is the disjoint union of its orbits under the action of G. The orbits whose orders are not divisible by p all have t elements. There are s such orbits.

2. *Let $G = \mathbf{S}_p$ be the symmetric group on p letters, where p is a prime number. Show that G has $(p-2)!$ p-Sylow subgroups and deduce the congruence $(p-1)! \equiv -1$ (mod p), which is known as Wilson's Theorem. Find the number of p-Sylow subgroups of $\mathbf{S}_{2p}$ if p is an odd prime (i.e., $p \neq 2$).*

An important point is that the number of elements of order p in a group is $p-1$ times the number of subgroups of order p in the group. Indeed, each subgroup of order p has $p-1$ elements of order p in it, and an element of order p is in a unique subgrop of order p, namely the group that it generates. In $\mathbf{S}_p$, there are $(p-1)!$ elements of order p. To see this, we think about the cycle decomposition of an element of order p and realize that the elements of order p in $\mathbf{S}_p$ are exactly the p-cycles. A p-cycle is described by a list $(a_1, \ldots, a_p)$, where the a_i are just the numbers from 1 to p, listed in some order. There are $p!$ lists, but each p-cycle is described by p different lists because lists should really be viewed as lying on a circle. Hence there are $(p-1)!$ elements of order p and thus $(p-2)!$ p-Sylow subgroups of $\mathbf{S}_p$. We get $(p-2)! \equiv 1$ mod p by the Sylow theorems; Wilson's theorem follows on multiplying by $p-1$. (See also `http://planetmath.org/encyclopedia/GroupTheoreticProofOfWilsonsTheorem.html`.)

The problem about $\mathbf{S}_{2p}$ looked harder than I intended, so I decided to take it out. On reflection, I decided that the best way to do the problem is to write down one p-Sylow subgroup P of $\mathbf{S}_{2p}$, calculate the order of the normalizer N of P, and then realize that the answer is the index of N in $\mathbf{S}_{2p}$. Take P to be the group of order p^2 that is generated by the two p-cycles $(1\,2\,\cdots\,p)$ and $(p\!+\!1\; p\!+\!2\,\cdots\,2p)$. The only p-cycles in P are the various non-trivial powers of the two generators. If σ conjugates P to P, then it has to map $(1\,2\,\cdots\,p)$ either to a power of itself or to a power of $(p+1\; p+2\,\cdots\,2p)$, and it has to map $(p+1\; p+2\,\cdots\,2p)$ either to a power of itself or to a power of $(1\,2\,\cdots\,p)$. There are two types of σ: those that map $\{1, \ldots, p\}$ to itself and those that map $\{1, \ldots, p\}$ to $\{p\!+\!1, \ldots, 2p\}$. Say σ is going to preserve $\{1, \ldots, p\}$ and will map each of $(1\,2\,\cdots\,p)$ and

$(p+1\, p+2 \cdots 2p)$ to powers of themselves. Then $\sigma(1)$ and $\sigma(2)$ can be arbitrary distinct elements of $\{1,\ldots,p\}$, but $\sigma(3),\ldots,\sigma(p)$ are determined by $\sigma(1)$ and $\sigma(2)$; similarly $\sigma(p+1)$ and $\sigma(p+2)$ determine $\sigma(p+3),\ldots,\sigma(2p)$. The number of σ that preserve $\{1,\ldots,p\}$ and that normalize P will be $(p(p-1))^2$; the same number of σ send $\{1,\ldots,p\}$ to $\{p+1,\ldots,2p\}$ and normalize P. This tells me that the order of N is $2(p(p-1))^2$ and thus that the number of p-Sylow subgroups is $(2p)!/(2p^2(p-1)^2)$. This number is 10 if $p=3$, for example. You can check easily that this number is 1 mod p, which it's supposed to be.

3. *If X is a subset of a group G, let $C(X)$ be the centralizer of X, i.e., the group of those elements of G that commute with all elements of X. Show that $C(X)=C(C(C(X)))$.*

This one turned out to be a matter of simple logic, with no deep mathematics involved. If X is a subset of G, then X commutes with all elements in $C(X)$, so X belongs to the commutator of $C(X)$: we have $X \subseteq C(C(X))$. Apply this with X replaced by $C(X)$; we get $C(X) \subseteq C(C(C(X)))$. Because X is in $C(C(X))$, the condition of commuting with $C(C(X))$ is at least as stringent as the condition of commuting with X, so that $C(C(C(X))) \subseteq C(X)$.

4. *Let G be a group whose order is twice an odd number. For $g \in G$, let α_g be the permutation of G given by the formula $x \mapsto gx$. Show that α_g is an even permutation if and only if g has odd order. Conclude that the elements of G with odd order form a subgroup H of G with $(G:H)=2$. Explain in your solution why it makes sense to talk about the sign of the permutation α_g; the potentially complicating issue is that G is not an ordered set.*

Let G be a group with n elements, and let g be an element of G. After we order the elements of G, multiplication by g (i.e., the map α_g) becomes a permutation σ of $\{1,\ldots,n\}$. If we re-order the elements of G, we change σ into a permutation of the form $\tau\sigma\tau^{-1}$, with $\tau \in \mathbf{S}_n$. Since σ and $\tau\sigma\tau^{-1}$ have the same sign, we can declare the sign of α_g to be the sign of σ; this number in $\{\pm 1\}$ will be well defined.

Suppose that g has order m; note that m divides n. The cycle decomposition of α_g reflects the decomposition of G into orbits under the action of the subgroup of G consisting of the powers of α_g. These orbits will have length m, and there will be n/m orbits. The sign of α_g is thus $\left((-1)^{m+1}\right)^{n/m}$. Hence α_g is an odd permutation if and only if m is even and n/m is odd.

In the situation of the problem, the order of G is twice an odd number. Hence if m is even, n/m is automatically odd. Thus the elements with even order are exactly those whose signs are odd; the elements of odd order are exactly those with even sign. (We declare the sign of g to be the sign of α_g.) The odd-order elements constitute the kernel of the map $G \to \{\pm 1\}$ defined by $g \mapsto \operatorname{sign} g$. Therefore, they form a subgroup of G.

5. *Let G be a group of order $2p$, where p is an odd prime number. Show that G is cyclic if and only if the 2-Sylow subgroup of G is normal.*

If G is cyclic, then it has a unique 2-Sylow subgroup: finite cyclic groups have at most one subgroup of a given order. Thus the 2-Sylow of G will be normal. Suppose now G has a normal 2-Sylow subgroup H. It will also have a normal p-Sylow subgroup P. (A subgroup of index 2 in a finite group is always normal.) It follows that $G=HP$ is isomorphic to the product $H \times P$. (Recall problem 13 of Chapter I, which appeared on the previous assignment.) The groups H and P are cyclic groups of co-prime order, so their product is again cyclic.

6. *Find two non-isomorphic nonabelian groups of order 30.*

How about $\mathbf{S}_3 \times \mathbf{Z}/5\mathbf{Z}$ and the dihedral group of order 30? The first group has only 3 elements of order 2, whereas the second group has 15 elements of order 2.

7. *Calculate the order of the conjugacy class of* $(1\,2)(3\,4)$ *in the symmetric group* $\mathbf{S}_n$ $(n \geq 4)$. *Find the order of the centralizer of* $(1\,2)(3\,4)$ *in* $\mathbf{S}_n$.

The conjugacy class consists of all products $(\sigma(1)\,\sigma(2))(\sigma(3)\,\sigma(4))$, where σ runs over $\mathbf{S}_n$. Thus the conjugacy class consists of all products of two disjoint transpositions. The number of such products is $n(n-1)(n-2)(n-3)/8$, if I'm not mistaken: $(a\,b)$ is the same as $(b\,a)$, and $(a\,b)(c\,d) = (c\,d)(a\,b)$. The centralizer of an element is a stabilizer under the action of $\mathbf{S}_n$ on itself by conjugation. I conclude that the centralizer of $(1\,2)(3\,4)$ has order $8n!/n(n-1)(n-2)(n-3) = 8(n-4)!$. This makes perfect sense, since the centralizer contains the symmetric group on $\{5, 6, \ldots, n\}$, the 4-cycle $(1\,3\,2\,4)$, and the 2-cycles $(1\,2)$ and $(3\,4)$.

8. *Suppose that G is a subgroup of the symmetric group $\mathbf{S}_n$ and that the order of G is a power of a prime number that does not divide n. Show that some element of $\{1, \ldots, n\}$ is left fixed by all permutations in G.*

The order of G has the form p^k, where p is a prime number. The assumption of the problem is that p does not divide n. (Thanks to Chu Wee for clarifying this on the comments page.) The group G acts on the set $S = \{1, 2, \ldots, n\}$. We are supposed to show that the set S^G is non-empty; here, S^G is the set of elements of S that are fixed by all elements of G. The congruence $\#(S) \equiv \#(S^G)$ mod p was established in class. The left-hand side, $\#(S) = n$ is prime to p, so the right-hand side, $\#(S^G)$, is non-zero mod p as well. Consequently, $\#(S^G)$ is non-zero, which means that S^G is non-empty.

9. *Suppose that G is a group with three normal subgroups N_1, N_2, N_3. Assume that $G = N_iN_j$ and that $N_i \cap N_j = \{e\}$ for $i \neq j$. Show that G is abelian and that the three normal subgroups are isomorphic to each other.*

As in the solution to problem 13 of Chapter I, elements of N_i commute with elements of N_j whenever i and j are distinct indices and $1 \leq i, j \leq 3$. Thus elements of N_1, for instance, commute with elements of $N_2N_3 = G$. Thus N_1 is in the center of G; so is N_2, by symmetry. Hence $G = N_1N_2$ is in the center of G, so that G is abelian. Now consider the map $N_1 \to G/N_2$ gotten by composing the inclusion $N_1 \hookrightarrow G$ and the canonical map $G \to G/N_2$. This map is an isomorphism of groups because $G = N_1N_2$ and $N_1 \cap N_2 = \{e\}$. Thus G/N_2 is isomorphic to N_1; by symmetry, it is also isomorphic to N_3. The conclusion now follows: N_1 and N_3 are isomorphic, and similarly N_1 and N_2 are isomorphic.

The last 8 problems came from old Math Department prelim exams, by the way.