Math 250A, Fall 2004
Problems due September 14, 2004

The problems this week were from Lang's "Algebra, Chapter I."

13. A key point here is that H and N have trivial intersection in part (a) because the order of the intersection divides the orders of both groups. For $h \in H$, $n \in N$, the commutator $nhn^{-1}h^{-1}$ lies in both groups: for example, it's in H because we can re-write it as $\left(nhn^{-1}\right)h^{-1}$ and the factor in parenthesis is in H because H is normal. Thus the commutators $nhn^{-1}h^{-1}$ are trivial, meaning that elements of H and N commute with each other. The set-theoretic map $H \times N \to HN$, $(h, n) \mapsto hn$ is then a homomorphism of groups. It's clearly surjective, and it's injective by problem 4. Part (b) looks to me like an inductive consequence of part (a), since H_r will be prime to the order of $H_1 \cdots H_{r_1}$ for each r.

14. Here, G is a finite group and N is a normal subgroup of G such that N and G/N have relatively prime orders. If H is a subgroup of G with the same order as G/N, then H and N have trivial intersection in G because $H \cap N$ has order dividing the gcd of the orders of H and N; this gcd is 1. Hence the order of HN is the product of the orders of H and N, i.e., of G/N and N. The product of these two orders is the order of G, so we must have $HN = G$.

For the second part of the problem, we first consider a homomorphism $f : N \to G$. The image of this homomorphism lies in N, since the composite of this homomorphism with the canonical map $N \to G/N$ has an image whose order divides both $\#(N)$ and $\#(G/N)$ and therefore must be trivial. In particular, if g is an automorphism of G, then the restriction of g to N maps N to N. Since this restriction is an injective homomorphism, and since N is finite, the restriction is surjective.

15–16. For each $s \in S$, let G_s be the stabilizer of s. We have to prove that the union of the G_s is not all of G. Now

$$\#(\bigcup_{s \in S} G_s) \leq \sum_{s \in S} \#(G_s);$$

on the other hand, $(G : G_s) = \#(S)$ for all s because there is only one orbit. Thus $\#(G_s) = \#(G)/\#(S)$ for each s, and we get $\sum_{s \in S} \#(G_s) = \#(G)$. We seem to get only the trivial inequality $\#(\cup_{s \in S} G_s) \leq \#G$ from all this. In fact, however, we have equality here only when the union of the G_s is a *disjoint* union. On the other hand, two G_s groups are never disjoint because they both contain the identity element of G. Conclusion: we have equality only when S is a singleton set. This possibility is ruled out by the hypothesis to the problem.

Next, consider problem 16 with $H \subset G$. We can introduce the G-set $S = G/H$ and observe that the stabilizer of a point $s = gH$ is gHg^{-1}. To say that the union of these stabilizers is not all of G—the assertion of problem 15—is to say that G isn't the union of the conjugates of H—the assertion of problem 16.

Another way to view problem 16 is to say that the number of conjugates of H is $G/N(H)$; here, we're thinking about the action of G on the set of conjugates of H by conjugation. The sum of the orders of the distinct conjugates of H is then $\#(H) \cdot \frac{\#G}{\#N(H)} = \#G \cdot \frac{\#H}{\#N(H)}$. This sum is less than the order of G unless H is its own normalizer; in that case, the sum is precisely the order of G. But even in this extreme case, the number of elements in the union of the conjugates of H can be equal to the sum of the orders of the conjugates only when the union is disjoint. If there are at least two conjugates, the union is not disjoint because all conjugates contain the identity element of G. To summarize: the union is conjugates can fill up G only when two conditions are satisfied: (1) H is its own normalizer, and (2) H has only one conjugate. Condition (1) implies that the number of conjugates of H is $(G : H)$. If this is true and H has only one conjugate, then $G = H$; this equality was excluded by the hypothesis.

17. This seems like a complete triviality. The set C is the union of its fibers relative to the projection map $C \to X$. In other words, for each $x \in X$, we can let C_x be the set of elements of C whose first coordinate is x. Then C is the disjoint union of the C_x and the number of elements of C is the sum of the numbers $\#(C_x)$. These numbers are the $\varphi(x)$ of the problem.

19. For each t in Gs, $Gt = Gs$. Hence the displayed sum in part (a) is just $1/\#(Gs)$ added up $\#(Gs)$ times. So, indeed, the sum is 1.

For part (b), it might be a good idea to introduce the subset C of $G \times S$ that consists of pairs (g, s) for which $gs = s$. By problem 17, the number of elements of C is the sum $\sum f(x)$ that appears in this part. If we project down to the second factor S instead of to the first factor G, we get that this sum is alternatively the sum $\sum_{t \in S} \#(G_t)$; here, G_t is the isotropy group of t: the set of g that fix t. Now the number of orbits for the action of G on S is $\sum_{t \in S} \frac{1}{\#(Gt)}$ in view of part (a) of this problem. On the other hand, $\frac{1}{\#(Gt)} = \frac{\#(G_t)}{\#G}$ for each t. Thus the number of orbits is $\frac{1}{\#G} \sum_{t \in S} \#(G_t)$, and we get the desired equality.

Summary: even though we haven't been doing anything deep here, we've actually proved a striking statement whose proof might be hard to find if we were starting from scratch: The number of orbits for the action of G on S is the average number of fixed points of an element of G. An equivalent (though more precise-sounding) statement is that the average number of fixed points is 1 on each orbit. Notice, by the way, that this implies the statement of problem 15. In that problem, the identity element has more than one fixed point on S; therefore, some element has to have 0 fixed points to make the average come out to be 1.

20. The group P operates on A by conjugation. Since P is a p-group, the number of fixed points for the operation is congruent mod p to the number of elements of A. Since A has p elements, the number of fixed points is therefore congruent to 0 mod p. Clearly, this number is at most p and at least 1 (because the identity is a visible fixed point). Hence there are p fixed points, meaning that the action by conjugation is trivial. Thus A is central.

22. Assume that H is normal in G and that H is a p-subgroup of G. (This means that the order of H is a power of p.) We have seen in class that H is contained in some p-Sylow subgroup P of G. Also, we have seen that all p-Sylow subgroups of G are of the form gPg^{-1} with $g \in G$. We see that gPg^{-1} contains gHg^{-1}, which is H. Hence H is contained in each p-Sylow subgroup of G.

23bc. Note that we did 23a in class. If $N(P) = N(P')$, then P' is contained in $N(P') = N(P)$, so $P' = P$ by part (a). For the final part, we note first that $N(P)$ is contained in $N(N(P))$: every subgroup of G normalizes itself! We have to show that the normalizer of $N(P)$ is contained in $N(P)$. This means that all elements of G that normalize $N(P)$ also normalize P. Suppose that $gN(P)g^{-1} = N(P)$, and consider gPg^{-1}, which is a p-Sylow subgroup of G that is contained in $N(P)$. Because P and gPg^{-1} are both p-Sylow subgroups of $N(P)$, they are conjugate in $N(P)$. This means that there is an $x \in N(P)$ such that $gPg^{-1} = xPx^{-1}$. But $xPx^{-1} = P$ because x normalizes P. Thus $gPg^{-1} = P$, so that g normalizes P, as required.