Math 250A, Fall 2004
Problems due September 7, 2004

I.4 *Let H and K be subgroups of a finite group G. Suppose that K is contained in the normalizer of H. Show that the number of elements in HK is $\#(H)\#(K)$ divided by the number of elements in the intersection of H and K.*

Consider the surjective map $\varphi : H \times K \to HK$ given by $(h, k) \mapsto hk$. The number of elements in $H \times K$ is $\#(H)\#(K)$. It suffices to show that the number of elements in $\#(H)\#(K)$ that map to a given element of HK is the number of elements in $H \cap K$. Suppose that we fix hk in HK. Then elements in $H \times K$ that map to hk have the form (h', k') where $h'k' = hk$. Write $h' = ht$; i.e., set $t = h^{-1}h'$. Then $k' = t^{-1}k$ and we find $t \in H$ and $t^{-1} \in K$. Thus t belongs to $H \cap K$. It is clear, more precisely, that the pairs (h', k') mapping to hk are in 1-1 correspondence with the elements t in $H \cap K$, the correspondence being given by $(h', k') \mapsto h^{-1}h' = kk'^{-1}$ and $t \mapsto (ht, t^{-1}k)$. This establishes the assertion of the problem. It is a well known fact that HK is a subgroup of G when K normalizes H, i.e., when K is contained in the normalizer of H. This is easy to check; the main point is that HK is closed under multiplication because we have

$$(h_1k_1)(h_2k_2) = h_1(k_1h_2k_1^{-1})k_1k_2$$

when the h_i are in H and the k_i are in K. Note that the element in parentheses on the right-hand side of the equation belongs to H because K normalizes H.

Another comment is that we can write HK as a disjoint union of cosets xK, where x runs over a set of representatives of the space $H/(H \cap K)$. This is more or less obvious. Indeed, HK is stable by right multiplication by K, so it is a union of various cosets xK with x in G. However, $xK \subseteq HK$ if and only if x can be chosen in H. Further, two cosets xK and yK (with $x, y \in H$) are equal if and only if x and y define the same class in $H/(H \cap K)$. Hence the number of elements in HK is the index $(H : (H \cap K))$ times the number of elements of K. This gives us another way to see the main formula of the problem.

I.5 *Let H be a subgroup of $G \times G'$, the product of two groups. Assume that the projections $p : H \to G$ and $p' : H \to G'$ are both surjective. Let N be the kernel of the second projection and let N' be the kernel of the first. [These are a priori subgroups of H, but is it slightly more convenient to think of them as subgroups of G and G', respectively. Thus the kernel of the second projection is more legitimately $N \times \{e'\}$ when we think of N as a subgroup of G.] Show that N is normal in G and that N' is normal in G'. Show that the image of H in $G/N \times G'/N'$ is the graph of an isomorphism from G/N to G/N'.*

To show that N is normal in G, take (n, e') in the kernel of the second projection and $g \in G$. By the surjectivity of the first projection, there is some g' in G' so that (g, g') belongs to H. The conjugate of (n, e') by (g, g') belongs to H; this conjugate is (gng^{-1}, e'), which is in the kernel of the second projection. Hence we have $gng^{-1} \in N$, which shows that N is normal. Symmetrically, we see that N' is normal in G'.

The group $N \times N'$ is now normal in $G \times G'$, and it is contained in H. Hence it is normal in H. The quotient $\bar{H} := H/(N \times N')$ is a subgroup of $(G \times G')/(N \times N')$, a group that we can identify with $G/N \times G'/N'$. There are two projection maps $\bar{H} \to \bar{G} := G/N$, $\bar{H} \to \bar{G}' := G'/N'$. These are surjective because H mapped surjectively to G and G'. To say that $\bar{H}$ is the graph of a homomorphism $\bar{G} \to \bar{G}'$ is to say that the first of these projection maps is an isomorphism. To say that $\bar{H}$ is the graph of an isomorphism $\bar{G} \xrightarrow{\sim} \bar{G}'$ is to say that both projection maps are isomorphisms. We need to show, therefore, that the projection maps are injective.

By symmetry, it's enough to discuss the first of the two. Consider the map $j : H \to \bar{G}$ gotten by composing the canonical map $H \to \bar{H}$ with the first projection $\bar{H} \to \bar{G} = G/N$. The kernel of j contains $N \times N'$, and the aim is to show that the kernel is precisely $N \times N'$. Suppose that h is an element of N in the kernel of j. Then the image of h in G is an element n of N. Let $k = (n, e') \in G \times G'$. Then k is an element of $N \times N'$ that has the same image as h in G. It follows that hk^{-1} has trivial image in G; it's therefore an element of N' (thought of as a subgroup of H). Hence h is the product of two elements of $N \times N'$—one element is in N and the second is in N'.

I.6 *Prove that the group of inner automorphisms of a group G is normal in* $\mathrm{Aut}(G)$*, the group of all automorphisms of G.*

For each $g \in G$, let ι_g be the inner automorphism $x \mapsto gxg^{-1}$. Let α be an automorphism of G. We must show for each g that the composite $\alpha\iota_g\alpha^{-1}$ is an inner automorphism of G. Computing, I found that $\alpha\iota_g\alpha^{-1} = \iota_{\alpha(g)}$. Indeed, if we apply $\alpha\iota_g\alpha^{-1}$ to $x \in G$, the result is $\alpha\left(g\alpha^{-1}(x)g^{-1}\right)$, which becomes $\alpha(g)\alpha\left(\alpha^{-1}(x)\right)\alpha(g^{-1})$. This simplifies further to $\alpha(g)x\alpha(g)^{-1}$.

I.7 Let G be a group such that $\mathrm{Aut}(G)$ is cyclic. Prove that G is abelian.

Let Z be the center of G: the group of elements of G that commute with all elements of G. The group Z is the kernel of the map $G \to \mathrm{Aut}(G)$ that takes each g to the inner automorphism "conjugation by g". Thus G/Z is naturally a subgroup of $\mathrm{Aut}(G)$. Since $\mathrm{Aut}(G)$ is cyclic, so is its subgroup G/Z. let g be an element of G whose image in G/Z is a generator of this cyclic group. Then each element of G may be written as a product $g^i z$ where i is an integer and z is an element of the center of G. Since g commutes with each z, it is clear that any two products $g^i z$ commute with each other. Hence G is abelian.

I.8 We have a group G and subgroups H and H' of G. (The groups H and H' could easily be the same subgroup of G in applications.) Define an equivalence relation $\sim$ on G by $x \sim y$ if and only if $y = hxh'$ for some $h \in H, h' \in H'$. You need to check that this really is an equivalence relation, which is not hard. Note also that the equivalence class of x is the set HxH', i.e., the double coset of x. Because an equivalence relation on a set partitions the set into disjoint equivalence classes, we get what is needed for part (a), namely that G is the disjoint union of the different HxH'.

Let $C = \{c\}$ be a set of representatives for the equivalence classes. (This is the notation introduced in part b with the addition that we've given a name to the set C.) This means explicitly: for each $x \in G$ there is a unique $c \in C$ so that there exist h and h' such that $x = hch'$. It is helpful to remember that h and h' are not necessarily unique. For example, imagine that G is abelian and that $H = H'$. Then we are writing $x = chh'$ with $h, h' \in H$. We might change h by multiplying it by an element t of H and change h' by multiplying it by t^{-1}. For each c, we consider $H \cap [c]H' = H \cap (cH'c^{-1})$ as in the text of the problem. This is a subgroup H_c of H; for example, in the case that we have contemplated (G abelian, $H' = H$), we have $H_c = H$. If $X_c = \{x_c\}$ is a set of representatives for the coset space H/H_c, then we do have $H = \coprod_{x_c \in X_c} x_c H_c$ as in the first of the two displayed formulas. This is nothing deep or exciting: we are simply writing H as a disjoint union of cosets of H_c in H.

Now G is the disjoint union over C of double cosets HcH'. I claim that a given HcH' is the disjoint union of cosets $x_c cH'$, indexed by the elements x_c of X_C. This claim is exactly what is needed to finish off the problem: it gives the displayed decomposition of G as a double disjoint union (where the first index is c, rather than x_c, as noted on the web page). Because c is fixed, we can drop the subscripts "c" from X_c and x_c: we'll call them X and x, respectively. (Thus x runs over X in the new notation.) To prove the equality $HcH \stackrel{?}{=} \coprod_x xcH'$ is to prove the equality $H \cdot [c]H = \coprod_x x([c]H)$; we pass between the two by multiplying by c or by c^{-1} on the right. Set $K = [c]H$. Then we need to know only that $HK = \coprod_x xK$, where x runs over a set of representatives of $H/(H \cap K)$. This decomposition was established above, at the end of our discussion of problem 1.

I.9 *Let G be a group and H a subgroup of G of finite index in G. Show that G contains a normal subgroup N of finite index in G such that N is contained in H.*

The group G acts on the finite set G/H by left translation; $g \in G$ sends the coset xH to gxH. This action is a group homomorphism $\alpha : G \to \mathrm{Perm}(G/H)$. If N is the kernel of α, then we have an inclusion $G/N \hookrightarrow \mathrm{Perm}(G/H)$. Especially, N has finite index in G because G/H is a finite set, so that $\mathrm{Perm}(G/H)$ is a finite group. (If G/H has n elements, $\mathrm{Perm}(G/H)$ has $n!$ elements.) The main issue now is that N consists of the elements of G that fix all cosets xH. Among these cosets is H itself, which is fixed precisely by the elements of H. (We have $gH = H$ if and only if g lies in H.) Thus we have $N \subseteq H$, and we get what was needed.

Suppose now that H_1 and H_2 have finite index in G. Let N_1 and N_2 be normal subgroups contained in H_1 and H_2 that have finite index in G. Then $N_1 \cap N_2$ has finite index in G. Indeed, this group is the kernel of the map $g \mapsto (gN_1, gn_2)$ from G to the finite group $G/N_1 \times G/N_2$. It follows that $H_1 \cap H_2$, which contains $N_1 \cap N_2$, has finite index in G as well.