Math 250A, Fall 2004
Homework Assignment #8
Problems due November 9, 2004

Chapter III, Problems 3, 9, 10, 11, 12, 13, 15, 17, 18

Note that 12(a) was assigned already in the previous homework. The group of ideal classes (or ideal class group) is defined on page 88.

Here are two further problems:

1. Let $\mathfrak{o}$ be a Dedekind ring, and let $\mathfrak{a}$ be a non-zero ideal of $\mathfrak{o}$. Suppose that α is a non-zero element of $\mathfrak{a}$, and let the prime decomposition of (α) be the product $\mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_t^{f_t}$ with $f_i \geq 1$. Show that we may write $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}$ with $e_i \geq 0$. Explain why there is an element of $\mathfrak{o}$ that belongs to $\mathfrak{p}_i^{e_i}$ but not to $\mathfrak{p}_i^{e_i+1}$, for $i = 1, \ldots, t$. If β is such an element, prove that $\mathfrak{a} = (\alpha, \beta)$ (the smallest ideal of $\mathfrak{o}$ that contains both α and β).

2. Suppose now that $\mathfrak{o}$ is a Dedekind ring that happens also to be factorial (Lang, p. 111). Prove that $\mathfrak{o}$ is a principal entire ring, thus showing that the converse of Theorem 5.2 on page 112 is true for Dedekind domains. Here is an outline of the proof; the exercise is to write up the proof, supplying details:

- Show that (π) is a prime ideal of $\mathfrak{o}$ if π is an irreducible element of $\mathfrak{o}$.
- If a is a non-zero element of $\mathfrak{o}$, show that (a) has a prime factorization $\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}$ in which the prime ideals $\mathfrak{p}_i$ are principal.
- If $\mathfrak{a}$ is an arbitrary non-zero ideal of $\mathfrak{o}$, show that $\mathfrak{a}$ has a prime factorization $\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}$ in which the $\mathfrak{p}_i$ are principal. Conclude from this that $\mathfrak{a}$ is principal.