Math 250A, Fall 2001
Homework Assignment #7
Problems due October 26, 2004

Part I: Dedekind Rings.

As we have discussed on the comments page, most authors define a Dedekind ring (or Dedekind domain) to be an integral domain that is Noetherian, of dimension 1 and integrally closed in its fraction field. Being of dimension 1 means that the domain is not a field and that non-zero prime ideals are maximal ideals. In homework #6, you showed that a Dedekind ring à la Lang is Noetherian in Exercise 13 and showed that a Dedekind ring is of dimension 1 in Exercise 18 (at the latest). It is understood in Exercises 13–19 that the ring $\mathfrak{o}$ is not a field; also, Lang has a tendency (or "convention," as he refers to it) to omit mentioning that ideals are non-zero. In the following additional problems, you will show that a Dedekind ring à la Lang is integrally closed in its fraction field.

1. Let $\mathfrak{o}$ be a Dedekind ring, and let K be the fraction field of $\mathfrak{o}$. Suppose that $\mathfrak{a}$ is a fractional ideal of $\mathfrak{o}$. Show that the set $\{\, x \in K \mid x\mathfrak{a} \subseteq \mathfrak{o} \,\}$ coincides with the fractional ideal $\mathfrak{a}^{-1}$.

2. Suppose now that $\mathfrak{O}$ (upper-case "$\mathfrak{o}$") is a subring of K that contains $\mathfrak{o}$ and is *finite* over $\mathfrak{o}$ in the sense that $\mathfrak{O}$ is a finitely generated $\mathfrak{o}$-module. (The action of $\mathfrak{o}$ on $\mathfrak{O}$ is by multiplication inside $\mathfrak{O}$.) Show that $\mathfrak{O}$ is a fractional ideal of $\mathfrak{o}$. Let $\mathfrak{f}$ be the fractional ideal $\mathfrak{O}^{-1}$, so that

$$\mathfrak{f} = \{\, x \in K \mid x\mathfrak{O} \subseteq \mathfrak{o} \,\}.$$

Show that $\mathfrak{f}$ is an integral ideal of $\mathfrak{o}$ and in fact that $\mathfrak{f}$ is even an integral ideal of $\mathfrak{O}$.

3. Using the inclusion $\mathfrak{f}\mathfrak{O} \subseteq \mathfrak{f}$, show that $\mathfrak{f} = \mathfrak{o}$ and then that $\mathfrak{o} = \mathfrak{O}$. Thus $\mathfrak{o}$ has the maximality property that it is equal to $\mathfrak{O}$ whenever $\mathfrak{O}$ contains $\mathfrak{o}$ and is finite over $\mathfrak{o}$.

4. Suppose that $a \in K$ is integral over $\mathfrak{o}$ in the sense that a satisfies an equation

$$a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0$$

with $c_0, c_1, \ldots, c_{n-1} \in \mathfrak{o}$. Show that $a \in \mathfrak{o}$. (Let $\mathfrak{O}$ be the smallest subring of K that contains $\mathfrak{o}$ and a.)

Part II: Fractional ideals in Dedekind rings are projective.

In the following problems, we let $\mathfrak{o}$ be a Dedekind ring.

5. Suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are integral ideals that are relatively prime. Show that the natural map $\mathfrak{a} \oplus \mathfrak{b} \to \mathfrak{o}$, $(x + y) \mapsto x + y$ is surjective and that its kernel is $\mathfrak{ab}$. Prove that the $\mathfrak{o}$-modules $\mathfrak{o} \oplus \mathfrak{ab}$ and $\mathfrak{a} \oplus \mathfrak{b}$ are isomorphic.

6. Suppose now that $\mathfrak{a}$ and $\mathfrak{b}$ are fractional ideals of $\mathfrak{o}$. (We do not assume that they are integral and we do not assume that they are relatively prime.) Prove that the two $\mathfrak{o}$-modules $\mathfrak{o} \oplus \mathfrak{ab}$ and $\mathfrak{a} \oplus \mathfrak{b}$ are isomorphic.

7. If $\mathfrak{a}$ is a fractional ideal, show that there is an $\mathfrak{o}$-module M so that $\mathfrak{a} \oplus M$ is a free $\mathfrak{o}$-module. Conclude that $\mathfrak{a}$ is a projective $\mathfrak{o}$-module.

Part III: one additional problem.

Do problem 7 on page 353, thus showing that rings that satisfy the standard Dedekind properties have the defining property à la Lang.