Math 250A, Fall 2004
Problems due September 21, 2004

1. Let G be a group of order $p^n t$, where p is prime to t and $n \geq 1$. As in class, we let S be the set of subsets of G having p^n elements and we view S as a G-set in the natural way. (If $X \in S$, gX is the set of gx with $x \in X$.) Recall from class that the size of the orbit of X is either exactly t or else is a multiple of p. Show that every orbit of size t contains exactly one p-Sylow subgroup of G and deduce the congruence $\#(S) \equiv st \bmod p$, where s is the number of p-Sylow subgroups of G.

2. Let $G = \mathbf{S}_p$ be the symmetric group on p letters, where p is a prime number. Show that G has $(p-2)!$ p-Sylow subgroups and deduce the congruence $(p-1)! \equiv -1 \pmod{p}$, which is known as Wilson's Theorem.

3. If X is a subset of a group G, let $C(X)$ be the centralizer of X, i.e., the group of those elements of G that commute with all elements of X. Show that $C(X) = C(C(C(X)))$.

4. Let G be a group whose order is twice an odd number. For $g \in G$, let α_g be the permutation of G given by the formula $x \mapsto gx$. Show that α_g is an even permutation if and only if g has odd order. Conclude that the elements of G with odd order form a subgroup H of G with $(G : H) = 2$. Explain in your solution why it makes sense to talk about the *sign* of the permutation α_g; the potentially complicating issue is that G is not an ordered set.

5. Let G be a group of order $2p$, where p is an odd prime number. Show that G is cyclic if and only if the 2-Sylow subgroup of G is normal.

6. Find two non-isomorphic nonabelian groups of order 30.

7. Calculate the order of the conjugacy class of $(1\,2)(3\,4)$ in the symmetric group $\mathbf{S}_n$ $(n \geq 4)$. Find the order of the centralizer of $(1\,2)(3\,4)$ in $\mathbf{S}_n$.

8. Suppose that G is a subgroup of the symmetric group $\mathbf{S}_n$ and that the order of G is a power of a prime number that does not divide n. Show that some element of $\{1, \ldots, n\}$ is left fixed by all permutations in G.

9. Suppose that G is a group with three normal subgroups N_1, N_2, N_3. Assume that $G = N_i N_j$ and that $N_i \cap N_j = \{e\}$ for $i \neq j$. Show that G is abelian and that the three normal subgroups are isomorphic to each other.