Suppose that N and H are subgroups of G and that H normalizes N. Assume that H and N have the trivial intersection. Then the map $H \times N \to HN$ given by $(h, n) \mapsto hn$ is clearly a bijection, in view of a problem from last week. However, it's not necessarily a homomorphism of groups because elements h and n need not commute inside of G. It's natural to write ${}^h n$ for the conjugate of n by h, namely hnh^{-1}; I put the exponent on the left because one tends to make conjugation into a left action.

A confusing aspect to this problem is that Lang talks about $H \times N$ in part (b) but uses $N \times H$ in part (c)! Since part (c) is the main part of the problem, let's try to write elements of N to the left of elements of H. Then if H normalizes N and $N \cap H = (e)$, the product $N \times H$ maps bijectively to $NH \subseteq G$, but we need to see what multiplication on $N \times H$ reflects that of NH. We have

$$n_1 h_1 n_2 h_2 = n_1 h_1 n_2 h_1^{-1} h_1 h_2 = n_1 {}^{h_1} n_2 \cdot h_1 h_2.$$

Hence, in $N \times H$, the second components multiply as usual but the first components multiply in a twisted way that isn't ridiculously far from what is written in the displayed equation of part (c) of this problem. The whole point is that we can work backwards. If we begin with groups H and N and a ψ as in the problem, we can produce a twisted group product $N \times H$ in which N is normal in such a way that the action of N on H by conjugation is given by ψ. To do this, we can help ourselves by introducing a short-hand for the image of n under $\psi(h)$: let's call this ${}^h n$. Then we define the twisted product on $N \times H$ by the formula

$$(n_1, h_1)(n_2, h_2) = (n_1 {}^{h_1} n_2, h_1 h_2).$$

If I haven't made a mistake, verification of the group axioms is now completely straightforward. One thing that needs to be checked is that hnh^{-1} in the group we've constructed is ${}^h n$. Of course, we write n for $(n, 1)$ and h for $(1, h)$; here, "1" just means the identity element of the relevant group. Thus, we want $(1, h)(n, 1) \stackrel{?}{=} ({}^h n, 1)(1, h)$. This is actually true; both products come out to be $({}^h n, h)$.