Math 250A — Professor K. A. Ribet

Last Midterm Exam — November 1, 2001

1 *(6 points). Establish the irreducibility over* $\mathbf{Q}$ *of each of the following polynomials:*

$$\begin{cases} x^{13} + 27x^2 - 120x + 69, \\ x^3 + 3x^2 + 9, \\ x^3 + x^2 + 2. \end{cases}$$

It might help to remember that we discussed three criteria for irreducibility in class.

The first polynomial is visibly an Eisenstein polynomial with $p = 3$. The second is irreducible mod 2. The third is irreducible if it has no roots in $\mathbf{Z}$, but the only possible integral roots are ± 1, ± 2. It's easy to check that these numbers aren't roots of the third polynomial.

2 *(6 points). Suppose that* A *is an integral domain (i.e., a commutative entire ring). Suppose that* I *and* J *are non-zero ideals of* A *for which the product* IJ *is a principal ideal. Show that the ideals* I *and* J *are finitely generated.*

See problem #13 on page 116, which was assigned for homework a few weeks ago. Suppose that $IJ = (a)$, and write $a = \sum_{i=1}^{n} x_i y_i$, with $x_i \in I$, $y_i \in J$. I claim that $I = (x_1, \ldots, x_n)$. For $x \in I$, we have $ax = \sum x_i(xy_i)$. Since $xy_i \in IJ = (a)$, we can write $xy_i = at_i$ for some $t_i \in A$. This gives $ax = \sum x_i a t_i = a \sum x_i t_i$. Then $x = \sum x_i t_i$ because A is entire. Therefore, $x \in (x_1, \ldots, x_n)$.

3 *(7 points). Find a set* X *and a subset* S_X *of* X *with the following property: if* A *is a set and* S *a subset of* A*, there is a unique map* $\varphi\colon A \to X$ *such that* $S = \varphi^{-1}(S_X)$.

Discuss the implication of the existence of (X, S_X) *for the association (sets)* $\to$ *(sets) that takes each set to the set of its subsets. (Explain how the association defines a contravariant functor and decide whether or not the functor is representable.)*

The set X that I had in mind is a set with two elements, say $X = \{0, 1\}$. We can then take $S_X = \{1\}$; what's important is that it be a 1-element subset of the set with two elements. Given $S \subseteq A$, we define $\varphi(a)$ to be 1 if $a \in S$ and 0 otherwise. Then clearly $S = \varphi^{-1}(S_X)$, and φ is the only map that works.

If A is a set, let $F(A)$ be the set of subsets of A. Then clearly $F(A) = \mathrm{Maps}(A, X)$ in view of what's in the previous paragraph. The point is that we can regard F as either a covariant or a contravariant functor. Indeed, if $f\colon A \to B$ is a map of sets and S is a subset of A, then $f(S)$ is a subset of B. But, in the other direction, if T is a subset of B, then $f^{-1}(T)$ is a subset of A. If we think of F as a covariant functor, then it isn't representable in any obvious way; my guess is that it isn't representable. (If you see why this is true, let me know.) On the other hand, if we think of F as a contravariant functor, then it's representable by X, together with the supplemental datum $S_X \in F(X)$. Namely, as discussed, we have for each A a bijection $F(A) \xleftarrow{\sim} \mathrm{Maps}(A, X)$ given by $\varphi \in \mathrm{Maps}(A, X) \mapsto \varphi^{-1}(S_X)$.

4 *(6 points). Consider the commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C \\ a\downarrow & & b\downarrow & & \downarrow c \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' \\ a'\downarrow & & b'\downarrow & & \\ A'' & \xrightarrow{u''} & B'' & & \end{array}$$

of abelian groups and homomorphisms.
Assume:

(1) *The kernel of v is the image of u;*
(2) *The kernel of b' is the image of b;*
(3) *The compositions $v' \circ u'$ and $a' \circ a$ are both 0;*
(4) *The maps c and u' are injective;*
(5) *The map a' is surjective.*

Show that u'' is injective. (Source: Bourbaki)

Take $\alpha'' \in A''$ with $u''(\alpha'') = 0$. Using the surjectivity of a', pick $\alpha' \in A'$ that maps to α''. Its image β' in B' is in the kernel of b', which is the image of b. Find $\beta \in B$ such that $b(\beta) = \beta'$, and let let $\gamma = v(\beta)$. The image of γ in C' is 0, because it's the image of β', which is the image of α' under u'. Because c is injective, $\gamma = 0$. By the exactness of the top row, there is an $\alpha \in A$ such that $\beta = u(\alpha)$. If we can show that $\alpha' = a(\alpha)$, then we are done because the image of α in A'' will be both α'' and 0. Let $\theta = a(\alpha)$. Then the commutativity of the diagram shows that the image of θ in B' is β', which is the same as the image of α'. But u' is injective, so $\theta = \alpha'$, as desired.