Mathematics 152 | Professor K. A. Ribet

Last Midterm Exam | October 26, 2010

Each problem is worth 5 points. You may consult a single sheet of notes and use a simple electronic calculator while taking this exam. Write carefully and clearly in complete English sentences; be sure to explain your reasoning at each step of your arguments.

1. A train makes a run of 120 miles. A second train starts one hour later and, traveling 6 mph faster than the first train, reaches the end of the run 20 minutes later than the first train. Assuming that each train travels at a constant speed, find the time needed by the slower train to make the run.

This problem appears on page 86 of the textbook. Let t be the time (in hours) taken by the slower train. Then $t - 2/3$ is the time taken by the faster train. The speeds of these trains are respectively $\frac{120}{t}$ and $\frac{120}{t-2/3}$ (measured in miles per hour). The difference between these speeds is 6; write this equation, clear denominators and examine the resulting equation. You should get $3t^2 - 2t - 40 = 0$. Using the quadratic formula, or otherwise, we see that t must be 4. (We reject the negative root of the quadratic.) The faster train takes 3 hours and 20 minutes; the speeds of the trains are 30 mph and 36 mph.

2. Show that the real polynomial $x^{100} + x - 1$ has exactly two real roots.

By Descartes's rules of signs, the polynomial has exactly one positive root. Change the sign of x and you see that the polynomial has exactly one negative root. Therefore it has exactly two real roots, since it's clear that 0 is not a root. According to sage, the roots are numerically -1.00699068585 and $+0.966583901079$.

3. Write the two square roots of $\frac{7}{25} + \frac{24}{25}i$ in the form $a + bi$.

The answer should be $\frac{4}{5} + \frac{3}{5}i$.

4. Let D the dilation of the plane with center $(0, 0)$ and scale factor 5. Let R be the map "reflection across the line $x = 10$." Describe the composite $R \circ D \circ R$ as a dilation (i.e., find its center and scale factor).

The scale factor is again 5, and the center is the point $(20, 0)$, i.e., the image of the origin under the reflection.

5. Find integers x, y and z so that

$$\log_a 2 = x \log_a \frac{10}{9} + y \log_a \frac{25}{24} + z \log_a \frac{81}{80}$$

for all $a > 1$.

This is a cute problem, which I got from the 19th century textbook that has the somewhat sketchy classic proof of Descartes's rule of signs that I mentioned in class. We have

$$\left(\frac{10}{9}\right)^x \left(\frac{25}{24}\right)^y \left(\frac{81}{80}\right)^z = 2.$$

Using unique factorization of primes, we get the three equations $x+2y-z=0$, $x-3y-4z=1$ and $-2x-y+4z=0$ by considering the powers of 5, 2 and 3 (respectively) on both sides of the equation. Solving these three equations in three unknowns by easy substitutions, we get $x=7$, $y=-2$, $z=3$.

6. Suppose that the circumcenter of $\triangle ABC$ is equidistant from the three sides of the triangle. Show that $\triangle ABC$ is equilateral.

This is problem 8 on page 230 of the textbook. Let x be the distance between the circumcenter O and each of the three points A, B and C. Let y be the distance between the circumcenter and each of the three sides of the triangle. If P is the midpoint of AB (say), then the triangle OPB is a right triangle whose hypotenuse has length x and whose leg OP has length y. The second leg PB has a length ($\sqrt{x^2-y^2}$) that we can calculate from the Pythagorean theorem. We see that the length of AB is then twice this number (i.e., $2\sqrt{x^2-y^2}$); we would get the same answer if we replaced AB by either of the other sides of $\triangle ABC$.