Mathematics 152 — Professor K. A. Ribet

Final Exam — December 15, 2010

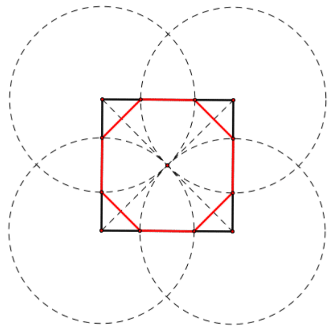
You may consult a single sheet of notes and use a simple electronic calculator while taking this exam. Write carefully and clearly in complete English sentences that explain your reasoning. The problems will be weighted roughly equally. (For problems 1–8, the maximum possible scores were: 6, 7, 6, 6, 6, 7, 6, 6, respectively, for a maximum possible total of 50.)

1. Suppose that we have five balls of distinct colors and a line of 12 boxes, numbered 1, 2, . . . , 12. Each box can hold no more than one ball. In how many distinct ways can the five balls be put in the 12 boxes? In how many distinct ways can the balls be put in adjacent boxes?

This problem is similar to a problem in the book, one that I think I assigned for homework. For the first question, one can choose first the 5 boxes that will receive the balls; there are $\binom{12}{5}$ ways to do this. For each of these choices, there are 5! ways of putting the balls into the chosen boxes. Thus the answer to the first question is $5!\binom{12}{5} = 95040$. For the second question, the adjacent boxes can be 1–5, 2–6, . . . , 8–12. Thus there are 8 ways to choose the boxes that will receive the balls. The answer here seems to be $8 \cdot 5! = 960$.

2. One web page offers the following construction of a regular octogon:

- Start with a unit square. (The sides of the square have length 1.)
- Find the center of the square by drawing the two diagonals and marking the point where they intersect.
- Draw the four circles whose centers are the four corners of the square and have radius equal to one-half the length of the diagonals.



- Note that each circle intersects the square in two points.

Verify that the eight points of the square that are cut by the circles do indeed form the vertices of a regular octogon. Find the length of each side of the octogon as well as the distance between the center and each of the eight points.

It might be helpful to have the square on the cartesian plane, with the vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$. The center will be at $(\frac{1}{2}, \frac{1}{2})$. The diagonals have length $\sqrt{2}$, so that the four circles have radius $\frac{\sqrt{2}}{2}$. The distance from each of the 8 distinguished points to the nearest vertex of the square is then $1 - \frac{\sqrt{2}}{2}$. Thus there is, for example, one vertex at $(1 - \frac{\sqrt{2}}{2}, 0)$ and another at $(\frac{\sqrt{2}}{2}, 0)$. The distance between these two points is $\sqrt{2} - 1$; that will be the length of each side of our octogon. We really need to check that all sides have the same length, so let's look at a slanted side, say the one between $(\frac{\sqrt{2}}{2}, 1)$ and $(1, \frac{\sqrt{2}}{2})$. If you use the formula for the distance between two points and do a bit of algebra, you'll get $\sqrt{2} - 1$, which is good! We want to see also that all 8 points have the same distance from the center. It's pretty clear that this will work out by symmetry; the distance is $\sqrt{1 - \frac{\sqrt{2}}{2}}$. The ratio $\frac{\sqrt{2} - 1}{\sqrt{1 - \frac{\sqrt{2}}{2}}}$ tells you what the length of the side of the octogon becomes if we scale things so that the distance from the center to the 8 vertices becomes 1, i.e., if we bring ourselves into the situation where the octogon is inscribed in the unit circle.

Do some more algebra and you'll get $\sqrt{2-\sqrt{2}}$. This is the right answer: we calculated the length of the side of a regular n-gon that's inscribed in the unit circle. The square of this length is $2-(\zeta+\zeta^{-1})$, where $\zeta = e^{2\pi i/n}$. For $n = 8$, we have $\zeta^4+1 = 0$, i.e., $\zeta^2+\zeta^{-2} = 0$. Since $(\zeta+\zeta^{-1})^2 = \zeta^2+\zeta^{-2}+2$, we get $(\zeta+\zeta^{-1})^2 = 2$ and then $\zeta+\zeta^{-1} = \sqrt{2}$.

3. Describe the graph of $9x^2+16y^2-54x+64y = -144$. Is it the empty set? an ellipse? a circle? a hyperbola? Find, as appropriate, the foci, semi-major axis, center, radius, asymptotes... of the graph.

Completing the square, you should get $9(x-3)^2+16(y+2)^2 = 1$ as the equation that we're graphing. The graph is a translation of the graph of $9x^2+16y^2 = 1$: we move 3 units to the right and 2 units down. We're dealing with an ellipse here. The foci are aligned horizontally; i.e., they are on the x-axis (before translation). The untranslated ellipse looks like the graph of the ellipse on page 104 of the book. We have $a^2 = \frac{1}{9}$, so that $a = \frac{1}{3}$; recall that a is the *semi-major axis*. The foci (before translation) are at $(\pm c, 0)$ where $\frac{1}{16} = a^2 - c^2$, so that $c^2 = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$. We have $c = \frac{\sqrt{7}}{12}$.

4. Prove carefully the following statement: *Suppose that*

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

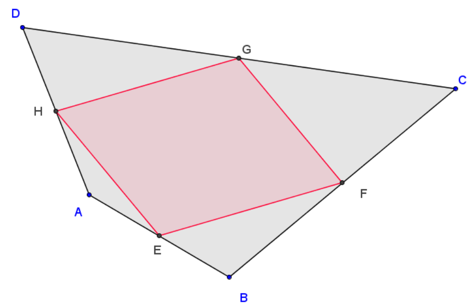
is a polynomial with integer coefficients. Let p *and* q *be integers with* $q \neq 0$ *and* $\gcd(p,q) = 1$. *Assume that f(p/q)=0. Then* p *divides* a_0 *and* q *divides* a_n.

This is called the "Rational Roots Theorem" in the book. The theorem is stated on page 184 and the proof is on the next page.

5. Suppose that you are given three points A, B and C that do not lie on a line. How you would construct, using straightedge and compass, a circle that passes through all three points? Is this circle unique?

Construct the perpendicular bisectors of AB and of AC. (You should allude to the method for doing this using ruler and straightedge.) Because A, B and C are not collinear, these bisectors are not parallel and therefore meet in a point O. Using the compass, draw a circle that's centered at O and has radius AO. This circle passes through B and C, as you'll explain. Yes, the circle is unique because the center will be equidistant from A and B and from A and C and therefore has to lie on both of the bisectors.

6. Let $ABCD$ be a convex quadrilateral and let E, F, G and H be the midpoints of the four sides of $ABCD$ (as shown). Prove that $EFGH$ is a parallelogram.



Draw the diagonal AC and examine $\triangle BAC$. Note that EF connects the midpoints of AB and of BC. By a baby version of FTS (proved last semester), AC and EF are parallel. Analgously, AC and HG are parallel. Hence HG and EF are parallel. Similarly for HE and GF.

7. Find an example of a non-zero polynomial $f(x)$ over a field F for which the associated polynomial function on F is 0; this means that $f(c) = 0$ for all numbers $c \in F$. Explain why there can be no such example if F is the field of rational numbers.

The simplest example of a polynomial whose associated function is 0 is the polynomial $x^2 - x = x^2 + x$ over the field $\{0, 1\}$ of integers mod 2. For the "Explain..." question, the essential point is that a non-zero polynomial has a degree, n; a polynomial of degree n can have no more than n roots (by the unique factorization theorem for polynomials), but the field of rational numbers is infinite and will therefore have more than n elements.

8. Suppose that a quadrilateral $ABCD$ is cyclic in the sense that its vertices lie on a circle. Using the fact that the angles of $\triangle BCD$ sum to 180 degrees, show that $|\angle A| + |\angle C| = 180$.

See page 281 of the textbook for the proof.