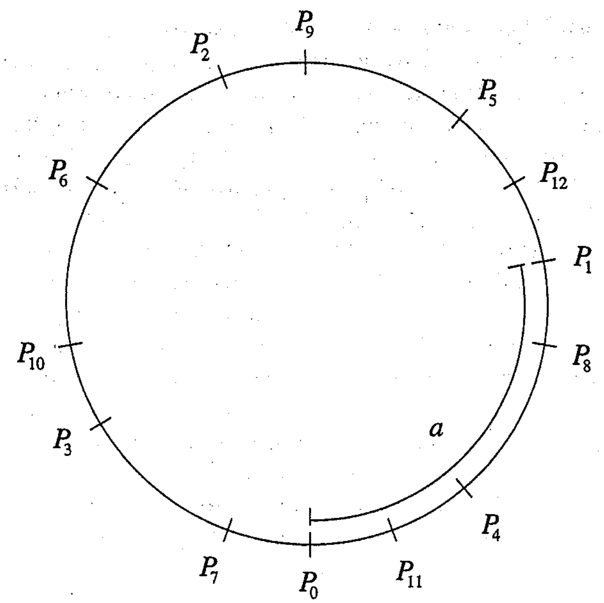

FIGURE 2.

$$a = rs, \quad b = tu, \quad c = rt, \quad d = su.$$

If, in addition, a and c are integers, then r may be taken to be an integer.

If $a = c$, then $b = d$ and we may choose $r = a$, $s = t = 1$, and $u = b$. If a and c are not equal, we assume that $a < c$ (otherwise, we may change the roles of a and c). We may interpret (1) in the following, geometric, way: Consider a circle of circumference c, and a point P_0 on it. We mark off arcs of length a, labeling the points in sequence P_1, P_2, through P_b. Equation (1) says that P_b coincides with P_0 and that we have gone around the circle a total of d times. If the tth point is the first that coincides with a previous one, then by the earlier observations, this point is P_0. The entire sequence returns to P_0 u times, and lands on each of the t distinct points $P_0, P_1, \ldots, P_{t-1}$ exactly u times, so

$$b = tu.$$

If we start at any point, say P_j, and mark off arcs of length a, we land on P_0 after $b - j$ steps, and continuing, we have $P_1, P_2, \ldots, P_{j-1}, P_j$, getting the entire sequence back. We can look at this in another way: We rotate the circle, putting P_0 in P_j's position, and starting at P_j we get the original sequence. This holds true for all points P_j, and the perimeter of the circle is therefore divided up into arcs of equal length by the t different points. Call this length r, and we have

$$c = rt.$$

Let s be the number of points on the arc between P_0 and P_1 in the positive direction, counting P_1 but not P_0. These points divide the arc into s equal parts of length r, and we have

$$a = rs.$$

Marking off the original sequence, we go around the circle d times. Every time we go around, we hit a point on the arc from P_0 to P_1 (again, not including P_0), hitting each a total of u times. This gives

$$d = su.$$

Thus we have shown the existence of the numbers r, s, t, and u. To verify the last claim, we observe that if the perimeter c and the arc length a are both integers, then every point is an integral number of arc lengths from P_0, and in particular, r is an integer.

Exercises:

13. Prove that the four number theorem remains true even if we remove the assumption that the numbers are positive.

14. Prove the four number theorem using induction on $|b|$ instead of geometry.

4. The questions raised in the introduction were related to the divisibility of integers. The four number theorem will play an important role in establishing fundamental relations for division.

It will be advantageous to define the concept of divisibility slightly differently from that of the ancient Greeks; in particular, we will consider the number itself to be one of its divisors. In accordance with this, we say that an integer a *is a divisor of* an integer b if there exists an integer c satisfying the equation

$$b = ac.$$

In this case, we also say that b *is a multiple of* a and that b is *divisible* by a. If a is a divisor of b, we express this symbolically[3] as $a \mid b$; otherwise, we write $a \nmid b$.

In the remainder of the book we will concern ourselves principally with integers, and unless we specify otherwise, all numbers should be considered as integers.

The properties of divisibility listed here follow easily from the definition. We will often use them without reference, and for that reason it is useful to see them once presented all together. The proofs are left to the reader. The letters used in the following relations represent arbitrary integers.

[3] Unfortunately, a symmetric symbol is used for an asymmetric relation; since this symbol is so widely accepted, we make no attempt to introduce another.