Math 114

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1 (5 points) Find the sum of the roots of $f(X) = X^9 - 5X^8 + 12X^7 - 6X^6 + 42$. Find the sum of the squares of these roots.

2 (11 points) Let $G = \mathbf{GL}(2, \mathbb{F}_p)$ be the group of two-by-two invertible matrices with coefficients in $\mathbb{F}_p$, under matrix multiplication.

(4 pts.) a. Suppose that $P \subseteq G$ is a p-Sylow subgroup of G and that $N(P) = \{ g \in G \mid gPg^{-1} = P \}$ is its normalizer. What general formula gives the number of p-Sylow subgroups of G in terms of the orders of G and of $N(P)$?

(4 pts.) b. Find a specific p-sylow subgroup P of G, and determine its normalizer.

(3 pts.) c. Sketch the method we used in class to calculate the order of G.

3 (11 points) Let $f(X) \in K[X]$ be an irreducible polynomial of degree n over a field K. Let $E = K[X]/(f(X))$.

(5 pts.) a. Prove that E is a field: explain how to find the inverse of a non-zero element of E.

(4 pts.) b. Suppose now that $K = \mathbb{F}_p$ is the field with p elements. Let G be the group of field automorphisms of E. What sort of group is G? (Is it finite? simple? abelian? Can its order be determined from p and n?)

(2 pts.) c. If $X^3 + aX + b$ is irreducible over $\mathbb{F}_p$, prove that $\Delta = -4a^3 - 27b^2$ has a square root in $\mathbb{F}_p$.

4 (7 points) Suppose that $f(X)$ is an irreducible polynomial of degree d over a field K. Let L be an extension of K of degree n, where $(n, d) = 1$. Show that $f(X)$ remains irreducible when considered as an element of $L[X]$. [Hint: use the Tower Law.]

5 (10 points) Let K be the field $\mathbb{Q}(e^{\frac{2\pi i}{n}})$, where n is a positive integer. Let α be a complex number whose nth power lies in K, and let $L = K(\alpha)$.

(5 pts.) a. Show that L/K is a Galois extension.

(5 pts.) b. Prove that $\mathrm{Gal}(L/K)$ is cyclic of order dividing n.

6 (10 points) Let N be a finite normal extension of K, and let E be a subfield of N which contains K. Set $n = [E : K]$, and denote by Σ the set of K-monomorphisms $E \to N$. Also, let $G = \mathrm{Aut}_K(E)$ be the group of K-automorphisms of the field E.

(3 pts.) a. What condition on the extension E/K is equivalent to the condition that Σ has exactly n elements?

(3 pts.) b. What condition on the extension E/K is equivalent to the condition $\Sigma = G$?

(4 pts.) c. Let F be the set of elements of E which are left fixed by all elements of G. Is E/F always a Galois extension?

7 (13 points) Suppose that $E \subset \mathbb{C}$ is the splitting field of a (non-constant) irreducible polynomial $f(X) \in \mathbb{Q}[X]$. Let G be the Galois group $\mathrm{Gal}(E/\mathbb{Q})$. Let α and β be roots of f in E, and set $\Omega = \{\, \sigma \in G \mid \sigma(\alpha) = \beta \,\}$.

(4 pts.) a. Is Ω necessarily non-empty?

(3 pts.) b. Is Ω a subgroup of G?

(3 pts.) c. How does the size of Ω vary if β changes and α stays fixed?

(3 pts.) d. Suppose that G is abelian. Prove that α is a real number if and only if β is a real number.

8 (8 points) Let L/K be a finite Galois extension, with Galois group G. For $x \in L$, set $\mathcal{N}\, x := \prod_{\sigma \in G} (\sigma x)$.

(4 pts.) a. Show $\mathcal{N}\, x \in K$.

(4 pts.) b. If L is a finite field, show that $\mathcal{N}$ is the map $x \mapsto x^i$, where $i = (L^* : K^*)$.

There are 75 points on this exam. Hope to see you again!