Mathematics 113 Professor K. A. Ribet

Last Midterm Exam October 31, 2013

Afternoon Edition

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences.* Your explanations are your only representative when your work is being graded.

The problems have equal weight.

1. Find the number of conjugates of $(1\,2\,3)(4\,5\,6)$ in A_6. (For this problem, and the ones below, be sure to explain your work in complete English sentences.)

In S_6, the conjugates of $\sigma = (1\,2\,3)(4\,5\,6)$ are precisely the permutations that have the same cycle type as σ. To make such a permutation, you lay down the six numbers $1 \to 6$ in some order. There are $6!$ ways to do this, but: each 3-cycle can be written in three ways, and the order of the two 3-cycles doesn't matter. Accordingly, there are $6!/(3 \cdot 3 \cdot 2) = 40$ such permutations. Still in S_6, the centralizer of σ thus has 18 elements. Among them is the permutation that swaps 1 with 4, 2 with 5 and 3 with 6. This permutation is $(1\,4)(2\,5)(3\,6)$; it's odd. Thus the situation is like many that we discussed in class. Namely, when we pass to A_6, the group order gets halved but so does the centralizer. Accordingly, σ has the same number of conjugates in A_6 as it does in S_6. Thus the answer is "40."

2. Let p be an odd prime, and let G be a dihedral group D_{2n}. Show that all p-Sylow subgroups of G are cyclic. Find the number of such subgroups.

There is a unique p-Sylow (which is therefore normal): it's the p-part of the cyclic group generated by r, which has order n. Specifically, write n as $p^i t$, where t is prime to p. Then the p-Sylow is the cyclic group generated by r^t, which has order p^i. [Note: if $i = 0$, one shouldn't technically speak of the p-Sylow subgroup of G because p-Sylows are supposed to be non-trivial. If you say that the number of p-Sylows is 0 in the case where p doesn't divide n, you'll get full credit and some extra respect.]

3. Suppose that G is a finite group and that H is a subgroup of G. Let $N = N_G(H)$ be the normalizer of H.

a. Let $H_1 = H, H_2, H_3, \dots, H_k$ be the distinct conjugates of H in G. Prove the formula

$$\sum_{i=1}^{k} |H_i| = |H| \cdot (G : N) = |G|/(N : H).$$

All the conjugates have the same number of elements, so the sum is $k \cdot |H|$. How do we know that $k = (G : N)$? It's a special case of the general rule that the orbit of $x \in X$

is G/G_x when G acts on a set X and G_x is the stabilizer of an element x of X. Here, X is the set of conjugates of H, and the orbit of H consists of the entire set (by definition). Now $(G : H) = (G : N)(N : H)$ (e.g., because all three indices can be written as fractions in a way that makes this formula obvious). Writing $(G : H) = |G|/|H|$, we get the equality of the middle expression and the expression on the right.

The takeaway here is that the sum on the left is $\leq |G|$ because the denominator $(N : H)$ is a positive integer.

b. If $H \neq G$, show that $\bigcup_{i=1}^{k} H_i \neq G$.

The sum of the sizes of the sets on the left is at most the size of G. Hence the union on the left can be all of G only if the union is disjoint. But the union isn't disjoint because 1 (the identity of G) is in all the groups H_i and because there are at least two groups H_i (in view of the assumption that H isn't all of G).

4. Let G be a group (possibly infinite) and let H be a subgroup of G for which the set G/H is finite. Use the action of G by left multiplication on G/H to show that there is a normal subgroup N of G such that $N \subseteq H$ and such that G/N is a finite group.

The indicated action gives you a homomorphism

$$\varphi : G \longrightarrow S_{G/H}.$$

Let N be the kernel of φ. We have $N \subseteq H$ because N is the group of elements of G that fix all elements of G/H, while H is the group of elements that fix the coset $H = 1 \cdot H$ in the set G/H. By the first isomorphism theorem, we have an injection $G/N \hookrightarrow S_{G/H}$. Since G/H is a finite set, the symmetric group $S_{G/H}$ is finite. Thus G/N is a finite group.

5. Let G be a group.

a. For each $g \in G$, let σ_g be the inner automorphism "conjugation by g." Suppose that φ is an automorphism of G. Establish the formula $\varphi\sigma_g\varphi^{-1} = \sigma_{\varphi(g)}$.

Let x be an element of G. We have

$$(\varphi\sigma_g\varphi^{-1})(x) = \varphi\left(g\varphi^{-1}(x)g^{-1}\right) = \varphi(g)\varphi(\varphi^{-1}(x))\varphi(g^{-1}) = \varphi(g)x\varphi(g)^{-1} = \sigma_{\varphi(g)}(x).$$

b. If G has trivial center and φ commutes with all σ_g, show that φ is the identity map.

By part (a), if φ commutes with all σ_g, then $\sigma_g = \sigma_{\varphi(g)}$ for all $g \in G$. Because G has trivial center, two elements a and b of G are equal if and only if the automorphisms σ_a and σ_b are equal. Indeed, if $\sigma_a = \sigma_b$, then you'll find by messing around that ab^{-1} commutes with all elements of G and is therefore the identity. Accordingly, we have $\varphi(g) = g$ for all $g \in G$, which shows of course that φ is the identity map.